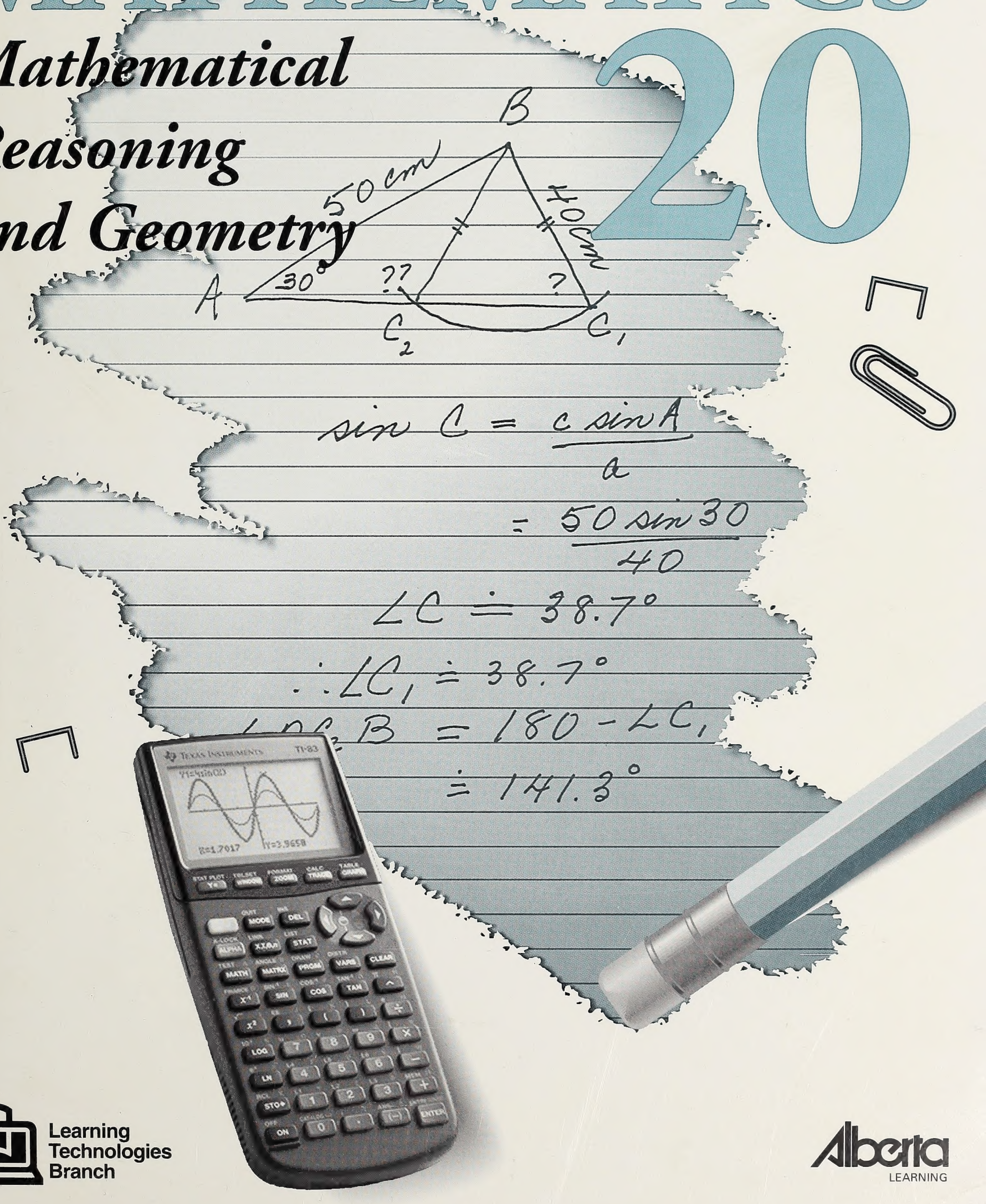
Pure MATHEMATICS 20

Module 5

Mathematical Reasoning and Geometry


B
50 cm
40 cm
A
30°
??
?
C_2
C_1
$\sin C = \frac{c \sin A}{a}$
$= \frac{50 \sin 30}{40}$
$\angle C \doteq 38.7°$
$\therefore \angle C_1 \doteq 38.7°$
$\angle C_2 B = 180 - \angle C_1$
$\doteq 141.3°$
TEXAS INSTRUMENTS TI-83

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Pure MATHEMATICS 20

Module 5

Mathematical Reasoning and Geometry

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Pure Mathematics 20
Student Module Booklet
Module 5
Mathematical Reasoning and Geometry
Learning Technologies Branch
ISBN 0-7741-1578-5

This document is intended for	
Students	✓
Teachers	✓
Administrators	
Parents	
General Public	
Other	

The Learning Technologies Branch has an Internet site that you may find useful. The address is as follows:

http://ednet.edc.gov.ab.ca/ltb

The use of the Internet is optional. Exploring the electronic information superhighway can be educational and entertaining. However, be aware that these computer networks are not censored. Students may unintentionally or purposely find articles on the Internet that may be offensive or inappropriate. As well, the sources of information are not always cited and the content may not be accurate. Therefore, students may wish to confirm facts with a second source.

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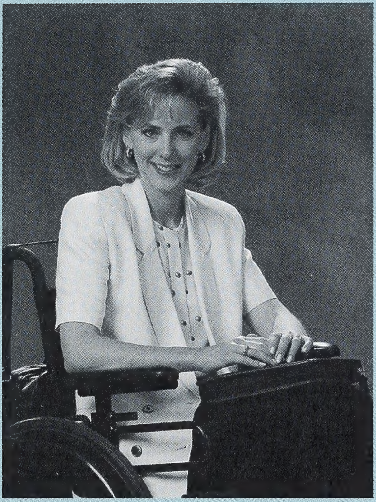
Welcome



Welcome to Module 5. We hope you'll enjoy your study of Mathematical Reasoning and Geometry.

Pure Mathematics 20

Module 1
Consumer Mathematics

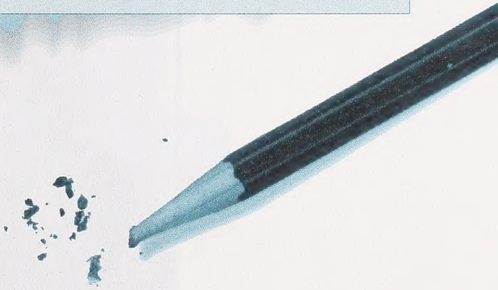
Module 2
Equations and Inequalities

Module 3
Quadratics

Module 4
Functions

Module 5
Mathematical Reasoning and Geometry

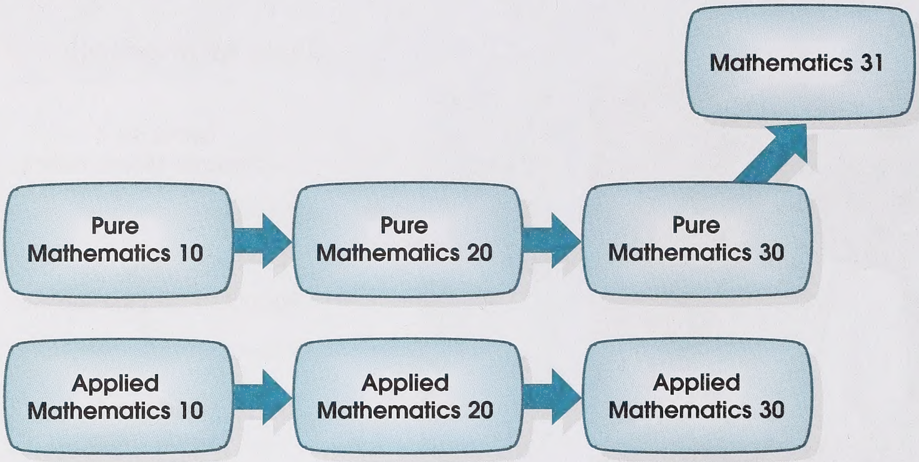
Module 6
Coordinate Geometry and Trigonometry



Pure Mathematics 20 contains six modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.

High School Mathematics Courses

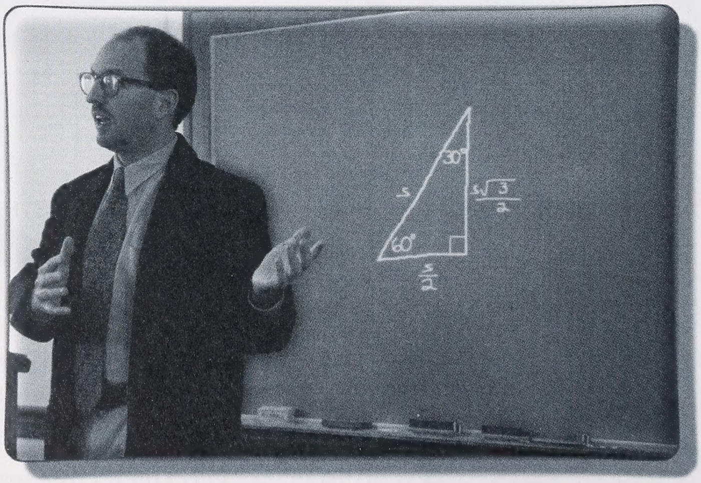
Pure Mathematics 20 is the second course in the Pure Mathematics 10–20–30 sequence of courses. Many students who take the Pure Mathematics 10–20–30 will also choose to take Mathematics 31. Another sequence of courses is Applied Mathematics 10–20–30.



How Do the Sequences Differ?

Each sequence of courses is designed for students with different mathematical strengths and interests. The Pure Mathematics 10–20–30 sequence is intended for students who are strong in mathematical theory and algebra. The Applied Mathematics 10–20–30 sequence is better suited to students who prefer to solve mathematical problems using numerical reasoning or geometry.

Each sequence of courses is designed for students with different career plans. Students who plan to take courses such as commerce, engineering, physics, and engineering technologies in post-secondary education will need to take Pure Mathematics 30 in high school. Some of these students will also need to take Mathematics 31.

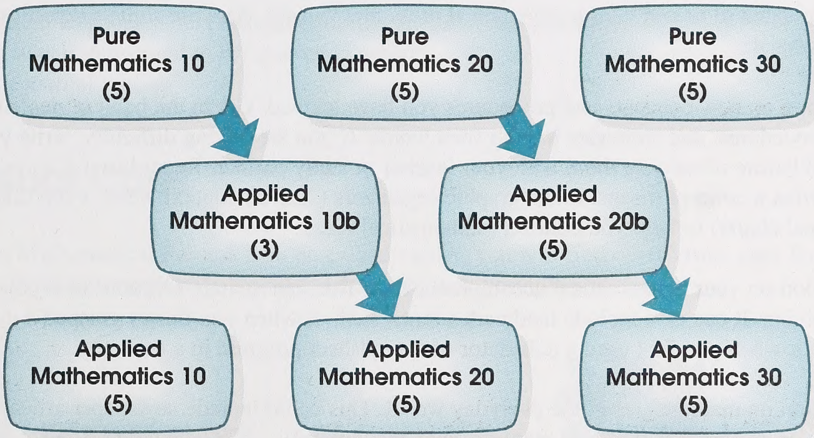


Can I Transfer to the Applied Mathematics Sequence of Courses?

You should be aware that the applied and pure mathematics courses do have some topics in common; other topics are specific to either applied or pure mathematics courses.

Applied Topics	Common Topics	Pure Topics
• linear programming • data tables and trends • design and layout • metric and imperial measure • data presentation • vectors and matrices • periodic, fractal, and recursive patterns • financial decision making • costing and design problems	• spreadsheets • line segments and linear graphs • scaling • triangles • surveys • financial mathematics • quadratic functions • circle geometry • the bell curve	• irrational numbers • exponents • polynomial and rational expressions • mathematical expectation • growth patterns • linear and non-linear systems • operations on functions • mathematical reasoning • exponential and logarithmic functions • conics • combinations • trigonometric functions

If you want to transfer from the Pure Mathematics 10–20–30 sequence to the Applied Mathematics 10–20–30 sequence at a future time, you won't have to repeat the topics that are common to pure mathematics and applied mathematics. If you decide to transfer to the Applied Mathematics sequence of courses after successfully completing Pure Mathematics 10, you may take the 3-credit course called Applied Mathematics 10b or you may take the 5-credit course called Applied Mathematics 10. If you decide to transfer after successfully completing Pure Mathematics 20, you may take the 5-credit course called Applied Mathematics 20b. The two bridging courses are shown in the following diagram:


Pure Mathematics 10 (5)
Pure Mathematics 20 (5)
Pure Mathematics 30 (5)
Applied Mathematics 10b (3)
Applied Mathematics 20b (5)
Applied Mathematics 10 (5)
Applied Mathematics 20 (5)
Applied Mathematics 30 (5)

Mathematical Processes

Throughout the course you will be expected to do the following:

- communicate mathematically
- connect mathematical ideas to everyday experiences and to concepts in other disciplines
- develop and use estimation strategies
- develop and use mental math strategies
- develop and use problem-solving strategies
- reason and justify your thinking
- select and use appropriate technologies as tools to solve problems
- use visualization to assist in processing information, making connections, and solving problems

Mathematical Process

■ Communication
■ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
■ Technology
■ Visualization

This icon shows which mathematical process you are expected to perform.

In order to develop these mathematical processes more fully, you are encouraged to ask someone who is also taking Pure Mathematics 20 to be your study partner. You will find that having a friend to discuss mathematics with will make your studying more enjoyable.

You are also encouraged to record your mathematical ideas in a journal. Here are some suggestions for organizing your journal:

- Keep a section on new concepts and procedures you have learned. Get in the habit of describing new concepts, procedures, and strategies in your own words. If you are having difficulty, write your questions in your journal before discussing them with your teacher or study partner. Record useful ways to help you remember what a concept means. Make graphic organizers (such as context webs, Venn diagrams, or organizational charts) to help you connect mathematical ideas.

- Keep a section on your mathematical accomplishments. This can include solutions to problems that you are proud of solving. It can also include landmark events, such as when you finally grasped a difficult concept (an "aha!") or when you first used a calculator or spreadsheet program in a new way.

- Keep a section on mathematics in the everyday world. This could include newspaper articles, cartoons, and descriptions of careers that require a mathematical background.

Resources You Will Need

In order to complete Pure Mathematics 20 successfully, you will need to have access to the following resources, which can be purchased through the Learning Resources Distributing Centre:

- the *MATHPOWER™ 11* textbook, Western Edition, published by McGraw-Hill Ryerson (1999)

- a graphing calculator

  Any of the following graphing calculators may be used: *Texas Instruments* TI-83, TI-83 Plus, TI-82, TI-86, TI-89, TI-92, or TI-92 Plus; *Casio* CFX-9850Ga-Plus, CFX-9850G, CFX-9800G, or FX-9700 series; or *Sharp* EL-9600c, EL-9600, EL-9300, or EL-9200. **Note:** Some of these calculators are no longer commercially manufactured.

- a computer and a spreadsheet program

  Any of the following spreadsheet programs may be used: *ClarisWorks™ 4.0* (or higher) for Macintosh, *ClarisWorks™ 4.0* (or higher) for Windows, *Microsoft® Works 3.0* (or higher) for Windows, or *Microsoft® Access 97* and *Microsoft® Excel 97* for Windows (a part of *Microsoft® Office Professional 95* or *Microsoft® Office Professional 97*).

Note: Examples in this course show the TI-83 graphing calculator and the *ClarisWorks™ 4.0* spreadsheet program where applicable. If you have access to a videocassette recorder, you may wish to view the video *The TI-83 Graphing Calculator Video Tutor* to review some of the calculator's features. Check with your distance learning provider concerning the availability of this video.

Finally, you will need the following resources, which can be purchased locally:

- a mathematical instrument set (compass, protractor, ruler, and triangles)
- a binder in which to respond to the questions asked
- a binder for journal writing

Other Visual Cues

In addition to the Mathematical Process icon described earlier, you will find other visual cues throughout this course. Read the explanations given to discover what each icon prompts you to do.

Refer to the textbook.

Use the Internet to explore a topic.

Answer the questions in the Assignment Booklet.

Remember: *Any Internet website address given in this module is subject to change.*

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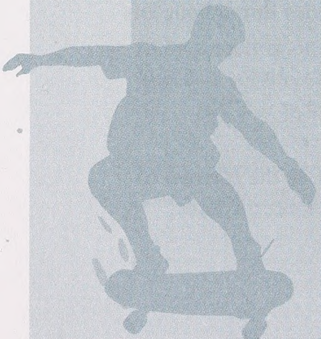

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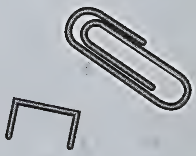
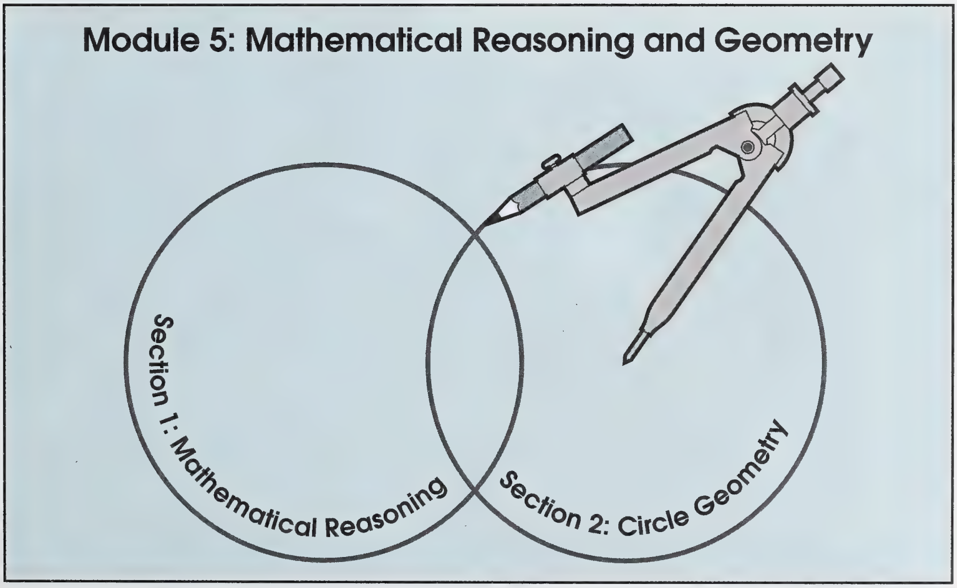


Module Overview

Do you like drawing with a compass and a straightedge? Do you find enjoyment in taking a clean sheet of paper and creating intricate geometrical designs? Just a few years ago, people were captivated by a toy that drew circles, spirals, and complicated curves. It is not difficult to imagine how the practical geometry of the ancient Egyptians and Babylonians evolved into a study of shape and structure for its own sake. Thanks to ancient Greek mathematicians, such as Euclid (ca. 300 B.C.), geometry was placed within a logical structure that emphasized reasoning and proof. Today, modern mathematicians continue this exploration.

In this module, you will explore mathematical reasoning and proof. You will make conjectures and investigate how you can argue if they are true or how you can reject them if they are false. You will then apply these logical processes to your study of geometry involving circles and polygons.


Module 5: Mathematical Reasoning and Geometry

Evaluation

The document you are presently reading is called a Student Module Booklet. It will show you, step by step, what to do and how to do it.

This module, Mathematical Reasoning and Geometry, has two sections. Within each section, your work is grouped into activities. Within the activities, there are readings and questions for you to answer. By completing these questions you will construct your own learning, discover mathematical connections, and practise or apply what you have learned. The suggested answers in the Appendix of this Student Module Booklet will provide you with immediate feedback on your progress.

At several points in this module you will be directed to the accompanying Assignment Booklet. In this module you are expected to complete two section assignments and one final module assignment. Your grading in this module is based upon the assignments that you submit for evaluation. The mark distribution is as follows:

Section 1 Assignment	**40 marks**
Section 2 Assignment	**40 marks**
Final Module Assignment	**20 marks**
TOTAL	**100 marks**

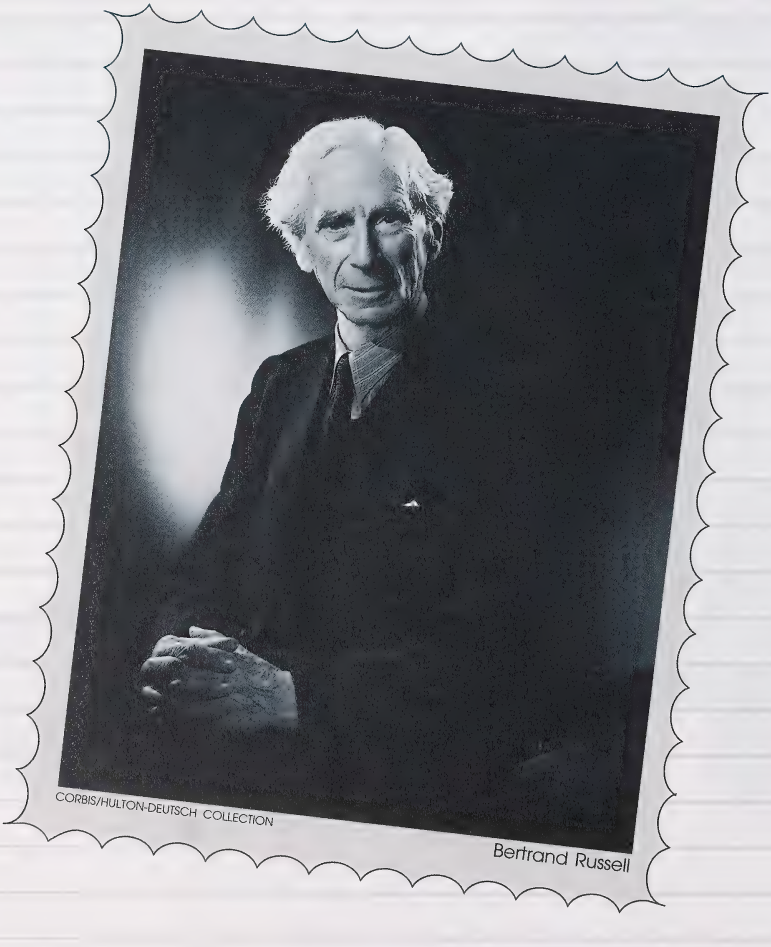
Strategies for Completing a Module

Before you begin, organize your materials: Student Module Booklet, textbook, Assignment Booklet, binder, lined paper, graph paper, graphing calculator, pens, pencils, and so on. Make sure you have a quiet area in which to work, away from distractions. Set up your time schedule.

To achieve success in this module, be sure to read all your instructions carefully and work slowly and systematically through the material. Remember, it's the work you do in this Student Module Booklet that will prepare you for your assignments and final test. Try to set realistic goals for yourself each day; and once you've set them, stick to them. Submit your assignments regularly, and don't forget to review your work before handing it in. Careful work habits will greatly increase your chances for success in Pure Mathematics 20.

Good luck!

Section 1



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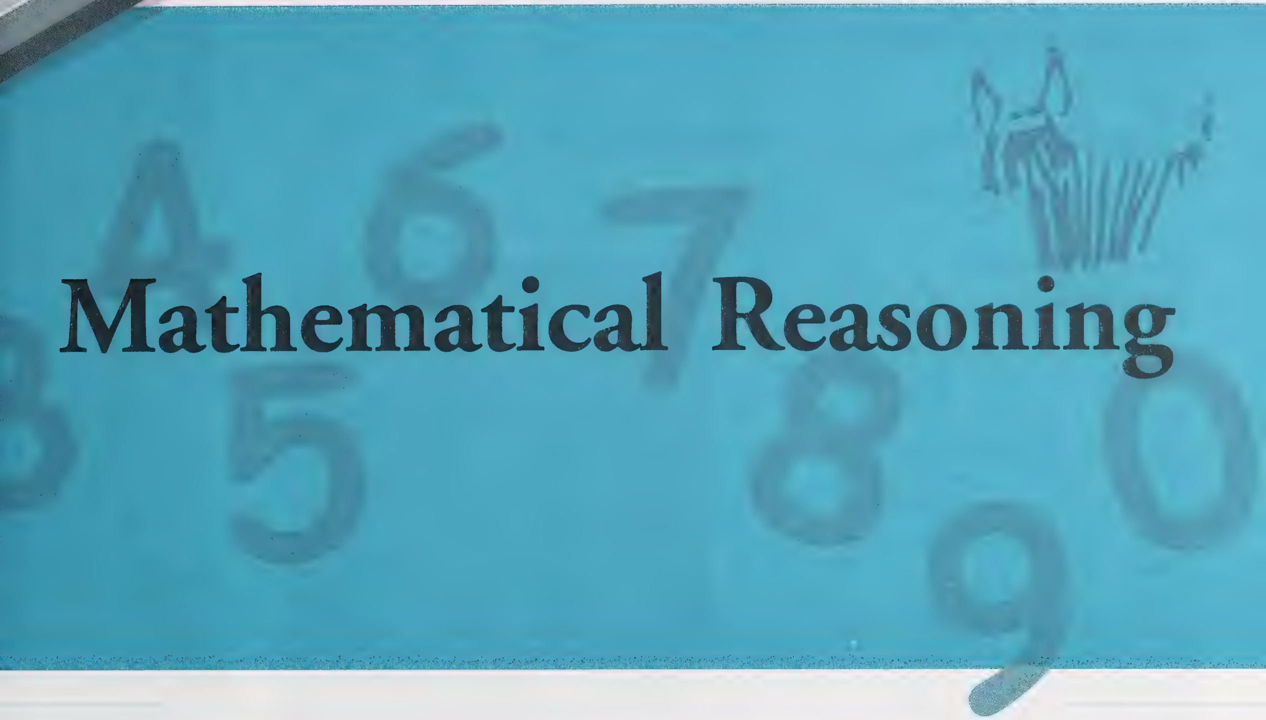
Bertrand Russell

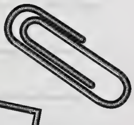


Mathematical Reasoning

Many people argue that the study of mathematics promotes the development of logical thinking. They feel certain the methods of reasoning underlying mathematics will assist students to solve a variety of problems systematically and draw valid conclusions about the world around them, even after much of the subject matter is forgotten.

Bertrand Russell (1872–1970) was one of the twentieth century's most eminent thinkers in mathematics, logic, and philosophy. His contributions included attempts to demonstrate that all mathematics can be derived from logical principles, particularly in his book titled *Principles of Mathematics*. Not only was he a great thinker, he was a social activist and pacifist who argued and demonstrated for world peace, disarmament, and the destruction of nuclear weapons.

In this section, you will explore the mathematical reasoning that Bertrand Russell helped clarify. You will investigate inductive, deductive, and logical reasoning; and you will be introduced to mathematical proof. Your study will help answer the question, "How can the truth of mathematical conjectures be demonstrated?"



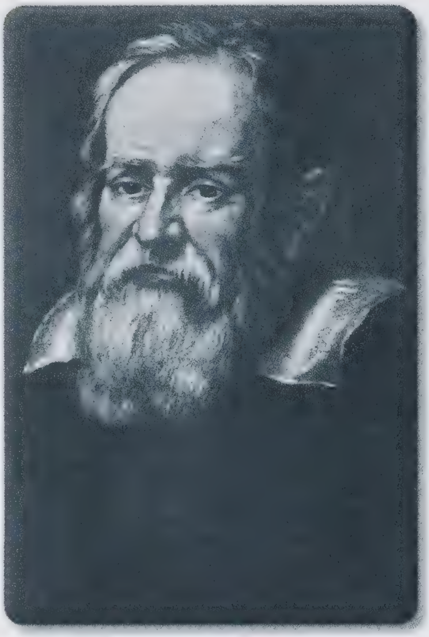
Activity 1: Inductive Reasoning and Conjecturing

Mathematical Process

☐ Communication
☒ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☐ Reasoning
☐ Technology
☐ Visualization

Prior to the Renaissance, most people believed that heavy objects fell faster than light objects. They argued that all you have to do is drop a feather and a rock at the same time to see which strikes the ground first. Today, people know that a feather and a rock, for example, only fall at different rates because of air resistance. In fact, a feather and a rock will fall at the same rate if they are released simultaneously in a vacuum (such as space).

One of the first scientists to conduct experiments with falling bodies was Galileo Galilei (1564–1642). He demonstrated, using inclined planes, that all objects, independent of mass, fall a distance proportional to the square of the time. Galileo's experiments could be performed by other investigators so they, too, could observe the same results.



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Galileo Galilei

The type of reasoning Galileo used to formulate his laws of mechanics was **inductive reasoning**. Inductive reasoning is based on patterns in observed data. These patterns are used to draw conclusions or make generalizations about the data.

inductive reasoning: a type of reasoning by which generalizations are drawn from patterns in observed data

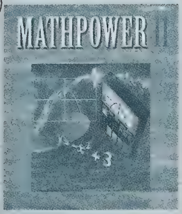
In addition to his work in mathematics and mechanics, Galileo studied lenses and light, invented the microscope, built telescopes to observe celestial bodies, and embraced the Copernican Theory—the theory that the Sun is the centre of the solar system and that Earth is just one of the planets that revolves around it.

If you have access to the Internet, you can learn more about Galileo at the following website:

http://galileo.imss.firenze.it/museo/b/egalilg.html

In this activity, you will reason inductively and make **conjectures** in much the same way that Galileo did concerning falling objects.

conjecture: a generalization made through inductive reasoning



Turn to page 338 of *MATHPOWER™ 11* and read "Inductive Reasoning and Conjecturing."

1. Answer the following questions on page 338 of the textbook:

- **a.** questions a. to e. of "Explore: Perform an Experiment"
- **b.** questions 1 to 7 of "Inquire"

Compare your responses with those in the Appendix, Section 1: Activity 1.

Some of the generalizations or conjectures you will be making can be proven through logical reasoning. A general conclusion that is first proven and then accepted as true is called a **theorem**.

theorem: a generalization that must be proven before being accepted as true

Mathematical Process

☒ Communication
■ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
■ Technology
■ Visualization

You may find it helpful to have a list of the theorems and definitions you encounter in this section so you can refer to them when problem solving, especially when you begin your study of geometry later in this module. Reserve a section in your journal for this list. Begin by writing the Triangle Angle Sum Theorem and the Exterior Angle Theorem from page 339 of the textbook.

You will now practise making conjectures.

Turn to pages 339 and 340 of *MATHPOWER™ 11* and work through Examples 1 to 3.

2. Answer the following questions on pages 340 to 342 of the textbook:

- **a.** questions 1, 2, 7, 8, 9, and 10 of "Practice"
- **b.** questions 11, 12, 16, 21, and 22.a. of "Applications and Problem Solving"

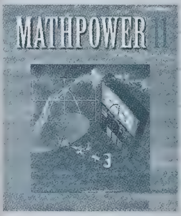
Compare your responses with those in the Appendix, Section 1: Activity 1.

As you know from experience, not all conjectures you or others make are necessarily true!

Although you may find many examples to support your conjecture, it only takes one example to make your conjecture false and, thus, disprove it. These examples are called **counterexamples**.

counterexample: an example that disproves a conjecture

Turn to page 343 of *MATHPOWER™ 11* and read "Analyzing Conjectures Using Examples and Counterexamples."



3. Answer the following questions on pages 343 and 344 of the textbook:

a. questions a. to c. of "Explore: Look for a Pattern"
b. questions 1 to 5 of "Inquire"

Compare your responses with those in the Appendix, Section 1: Activity 1.

Searching for counterexamples is all about not jumping to conclusions. Finding one or two examples that support your conjecture does not mean it is true in every situation.

Turn to pages 344 and 345 of *MATHPOWER™ 11* and work through Examples 1 to 3.

4. Answer the following questions on pages 345 and 346 of the textbook:

a. questions 1, 3, 4, 8, 9, and 10 of "Practice"
b. questions 13, 14, 16, and 20 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 1: Activity 1.

In the exercise you just completed, you worked with **acute triangles**, **obtuse triangles**, and **isosceles triangles**. Add these definitions to your list of theorems and definitions.

acute triangle: a triangle with three acute angles

obtuse triangle: a triangle with one obtuse angle

isosceles triangle: a triangle with two congruent sides

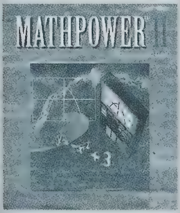
Now Try This

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ☒ Problem Solving
- ☒ Reasoning
- ■ Technology
- ■ Visualization

One approach to exploring a problem is to hypothesize, to make conjectures, to propose possible solutions, to probe, and to test. Even conjectures that are later discounted add valuable insight and can be synthesized with alternate suppositions that will eventually lead you to a solution. These conjectures help you define the problem and its limits and open possibilites. A written record of these conjectures (your tests and trials) will keep you from returning to barren ground; and when you revisit the problem, your record will jog your memory, reminding you of the progress you have made.

You will now put conjecturing to use in solving a non-routine problem. Keep a record of your thinking and your conjectures; it will help you if you need to look at the problem a second time.



5. Turn to page 346 of *MATHPOWER*™ *11* and answer questions 1 and 2 of "PATTERN POWER."

Compare your responses with those in the Appendix, Section 1: Activity 1.

Looking Back

Mathematical Process

- ☒ Communication
- ☒ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ☒ Reasoning
- ■ Technology
- ■ Visualization

In this activity, you explored inductive reasoning and conjecturing, and you used examples either to support your conjectures or to disprove them.

Get a copy of your local newspaper and identify articles containing examples of inductive reasoning. Critique the stated conjectures or those implied by the article. Can you find counterexamples refuting these conjectures? Record your findings in your journal and share them with another student taking Pure Mathematics 20.

Activity 2: Deductive Reasoning

CORBIS/BOB KRIST

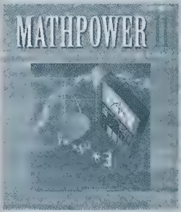
Have you read any of Sir Arthur Conan Doyle's Sherlock Holmes stories? The silhouette on the wall at the Baker Street subway station in London is immediately recognizable as that of the great detective. Few people can think of Sherlock Holmes without being reminded of his remarkable powers of deduction. From just a footprint in the soil, he could deduce the sex, height, weight, age, and peculiarities of physique and gait of the person who made it!

In this activity, you, too, will be using **deductive reasoning** to prove conjectures. You will begin by comparing and contrasting deductive and inductive reasoning.

deductive reasoning: a process of demonstrating that, if certain statements are true, then other statements follow logically from them

Mathematical Process

- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☐ Visualization



Turn to pages 347 of *MATHPOWER™ 11* and read "Deductive Reasoning."

1. Answer the following questions on page 347 of the textbook:

  a. questions a. to d. of "Explore: Use Inductive and Deductive Reasoning"
  b. questions 1 to 6 of "Inquire"

Compare your responses with those in the Appendix, Section 1: Activity 2.

In the preceding questions, you examined how inductive and deductive reasoning could be used to justify the Opposite Angles Theorem. Did you remember to include this theorem in your list?

Turn to page 348 of *MATHPOWER™ 11* and read from the top of the page to the red line on page 349, working through Examples 1 to 3.

2. Answer the following questions on page 349 of the textbook:

- **a.** questions 2, 4, 6, and 8 of "Practice"
- **b.** questions 9 to 13 and 18 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 1: Activity 2.

The Connecting Words *And*, *Or*, and *Not*

Before you look at more proofs using deductive reasoning, you need some additional background in logic.

statement: a sentence that is either true or false

In mathematics, every **statement** is a sentence that is either true or false; a statement is not debatable. Look at the photograph on the right; then consider the following statement, p:

p: "This bird is a Canada Goose."

Is this statement true or false? If the statement is true, you can change it into a false statement simply by inserting the word *not*.

$\sim p$: "This bird is not a Canada Goose."

negation: the opposite of a statement, formed using the word "not"

When you use the word *not* this way, the statement you form is called a **negation**. You can use the symbol $\sim p$, read "not p," to represent the negation. If the original statement is true, then its negation is false. If the original statement is false, then its negation is true. The truth values of a statement and its negation may be summarized in a table.

p	$\sim p$
T	F
F	T

Two statements can be linked together to form a **compound statement**. Consider the following statements, *p* and *q*:

p: "This bird is a Canada Goose."
q: "This bird is a gander."

Note: *A gander is a male goose.*

Now you can form a compound statement using the connecting word *and*.

p and q: "This bird is a Canada Goose, and this bird is a gander."

The compound statement *p and q* is only true if *p* and *q* are both true.

p	q	p and q
T	T	T
T	F	F
F	T	F
F	F	F

You can also form a compound statement using the connecting word *or*.

p or q: "This bird is a Canada Goose, or this bird is a gander."

The compound statement *p or q* is true if only *p* is true, if only *q* is true, or if both *p* and *q* are true. For example, the compound statement *p or q* is true, even if the bird is a female.

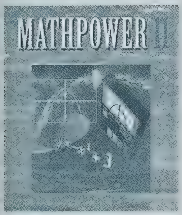
p	q	p or q
T	T	T
T	F	T
F	T	T
F	F	F

compound statement: two statements linked together with a connective, such as "and," "or," or "not"

Mathematical Process

☐ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☒ Reasoning
☐ Technology
☐ Visualization

You will now continue working with compound statements using *and* and *or*.

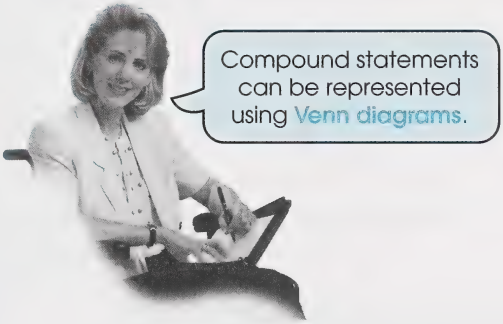


Turn to page 350 of *MATHPOWER*™ *11* and read “The Connecting Words *And*, *Or*, and *Not*.”

3. Answer the following questions on pages 350 and 351 of the textbook:

- **a.** questions a. to d. of “Explore: Compound Statements With *And* and *Or*”
- **b.** questions 1 and 2 of “Inquire”

Compare your responses with those in the Appendix, Section 1: Activity 2.


Compound statements can be represented using Venn diagrams.

Venn diagram: a diagram of overlapping circles drawn within a rectangle used to model relationships among statements

Mathematical Process

- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☒ Visualization

In the Venn diagrams on the right, statements p and q are represented by circles. The interior of each circle represents the statement or contains the elements defined by the statement. The exterior of the circle is the negation of the statement. For example, suppose statement p is “x is a positive factor of 6.” Then the numbers 1, 2, 3, and 6 would lie inside circle p and the numbers that are not factors of 6 would lie outside the circle.

The compound statement *p and q* is represented by the overlap of the two circles—the shaded portion in Diagram 2.

The compound statement *p or q* is represented by both circles—the shaded portion in Diagram 3.

Diagram 1

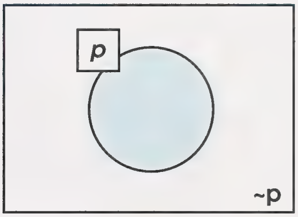


Diagram 2

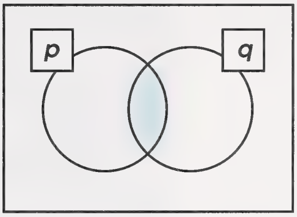
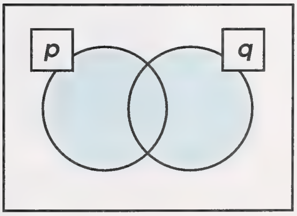


Diagram 3



Example

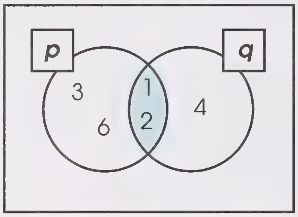
Use the following statements to answer questions a. and b.:

p: "*x* is a positive factor of 6."
q: "*x* is a positive factor of 4."

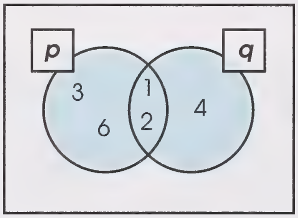
a. Draw a Venn diagram of *p and q*.
b. Draw a Venn diagram of *p or q*.

Solution

a. The elements defined by *p* are 1, 2, 3, and 6. The elements defined by *q* are 1, 2, and 4. The elements that are defined by *p and q* are the positive factors common to 6 and 4. There are two elements in the overlap: 1 and 2.



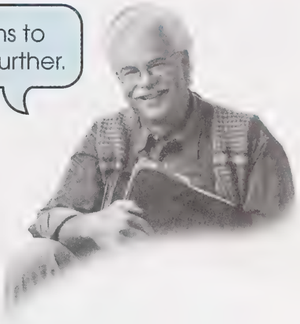
b. The elements that are defined by *p or q* are the positive factors of 4, or 6, or both. These elements are 1, 2, 3, 4, and 6.

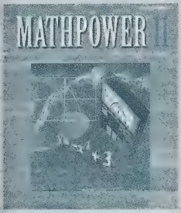


If you have access to the Internet, you can explore Venn diagrams further at the following website:

http://www.combinatorics.org/Surveys/ds5/VennEJC.html

Now, you will use Venn diagrams to explore compound statements further.





Turn to pages 351 to 354 of *MATHPOWER™ 11* and work through Examples 1 to 7.

4. Answer the following questions on pages 354 to 356 of the textbook:

  a. questions 1, 2, 6, 8, 12, 16, 18, 24, 31, 34, 36, 40, and 46 of "Practice"

  b. questions 52.b., 53, 54.a., 54.c., 56, and 57 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 1: Activity 2.

Conditional Statements—Converses and Contrapositives

conditional statement: an "If...then" statement

Many statements in mathematics are of the form, "If p, then q." For example, if ΔABC is scalene, then no two sides are equal in length. An *If...then* statement is called a **conditional statement** or an implication.

Mathematical Process
☒ Communication
☒ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
■ Technology
■ Visualization

Conditional statements occur frequently in everyday speech. Suppose you and your friends are planning to attend a late-evening outdoor concert. You look out the window and see an approaching storm. Your friend comments, "If it storms, then the concert is cancelled."

The statement in the *if* clause is called the **hypothesis**; and the statement in the *then* clause is called the **conclusion**.

hypothesis: "It storms."
conclusion: "The concert is cancelled."

There are related conditional statements that can be derived from the original conditional statement. One such statement is the **converse**. The converse is formed by interchanging the hypothesis and the conclusion. For example,

original conditional: "If it storms, then the concert is cancelled."
converse: "If the concert is cancelled, then it is storming."

The converse and the original conditional are not logically equivalent. You will agree that if the concert is cancelled, then the reason could be other than a storm; the concert could be cancelled if the performers went on strike!

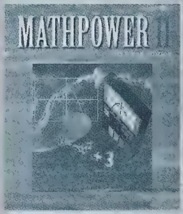
hypothesis: the part in the "if" clause of a conditional statement

conclusion: the part in the "then" clause of a conditional statement

converse: a conditional statement formed by interchanging the "if" and "then" clauses of another conditional statement

To discover more about conditionals, read on!

Turn to page 358 of *MATHPOWER™ 11* and read "*If…then* Statements, the Converse, and the Contrapositive."



5. Answer the following questions on page 359 of the textbook:

- **a.** questions a. to g. of "Explore: Interpret the Statement"
- **b.** questions 1 to 5 of "Inquire"

Compare your responses with those in the Appendix, Section 1: Activity 2.

Mathematical Process

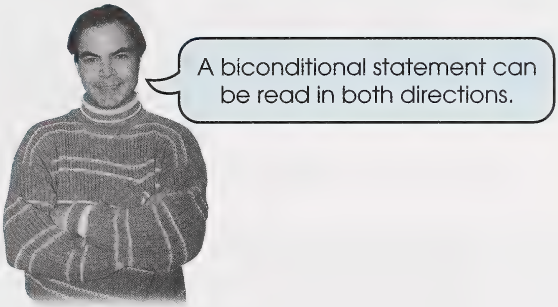
- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ☒ Reasoning
- ■ Technology
- ■ Visualization

biconditional statement: a statement formed by combining a conditional and its converse, using "if and only if"

contrapositive: a conditional statement formed by negating both the "if" and "then" clauses of a converse

As you discovered from the previous set of questions, the converse of a true conditional statement is not necessarily true. However, if the converse of a true conditional statement is also true, the conditional and its converse can be combined into an *if and only if* statement, a **biconditional statement**. For example,

conditional statement: "If $x+3=5$, then $x=2$."
converse: "If $x=2$, then $x+3=5$."
biconditional statement: "$x+3=5$ if and only if $x=2$."


A biconditional statement can be read in both directions.

Another conditional statement that can be derived from the original conditional statement is a **contrapositive**. A contrapositive can easily be formed by first writing the converse, then negating both the hypothesis and the conclusion.

Example

a. Form the contrapositive of the following conditional statement:

"If it storms, then the concert is cancelled."

b. Discuss the truth value of the contrapositive, if the original conditional statement is true.

Solution

a. **conditional statement:** "If it storms, then the concert is cancelled."
converse: "If the concert is cancelled, then it is storming."

Now, form the contrapositive by negating the hypothesis and the conclusion.

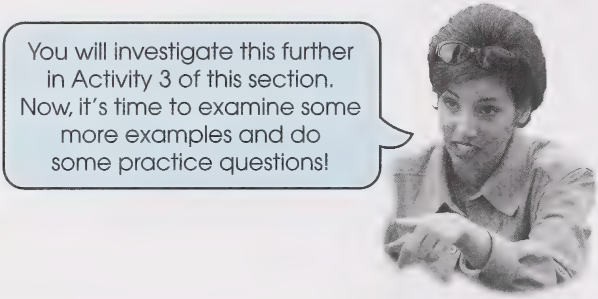
contrapositive: "If the concert is not cancelled, then it is not storming."

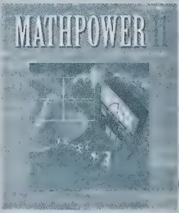
b. The contrapositive is also true. If they didn't cancel the concert, it can't be storming. This is because if it were storming and if you are to believe the original statement, then the concert would have been cancelled.

Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☐ Visualization

Mathematicians say the contrapositive and the original conditional are logically equivalent. They are either both true, or they are both false. One cannot be true and the other false. This property can be used to prove a conjecture. Instead of proving the original statement, you can prove its contrapositive. Once you have proven the contrapositive, you can conclude the original conjecture is true.


You will investigate this further in Activity 3 of this section. Now, it's time to examine some more examples and do some practice questions!



Turn to page 359 of *MATHPOWER™ 11* and read from the red line to the red line near the bottom of page 361, working through Examples 1 to 4.

6. Answer the following questions on pages 361 and 362 of the textbook:

a. questions 2, 3, 4, 9, 12, 17, 18, 19, 23, and 26 of "Practice"
b. questions 30 and 31 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 1: Activity 2.

Now Try This

Mathematical Process

- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

A couple of illustrations and examples in *MATHPOWER™ 11* are based on Lewis Carroll's *Alice's Adventures in Wonderland*. Lewis Carroll is the pen name of Charles Dodgson (1832–1898) who lived and taught mathematics in Oxford, England.

If you have access to the Internet, you can find out more about Lewis Carroll, logical reasoning, and his contributions to mathematics at following website:

http://www.lewiscarroll.org/logic.html

Looking Back

Mathematical Process

- ☒ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☐ Visualization

In this activity, you explored deductive reasoning. In particular, you investigated logical statements and compound statements formed using the connectives *and*, *or*, and *not*. You also considered conditional statements, their converses and contrapositives, and biconditional statements.

In your journal, write examples of each of the following:

- A true conditional statement and its true converse.
- A true conditional statement and its false converse.
- A false conditional statement and its true converse.

Activity 3: Direct and Indirect Proof

proof: a convincing argument

direct proof: a proof in which you begin with an assumption and work deductively towards a valid conclusion

indirect proof: a process of arguing in which you assume the conclusion you wish to demonstrate is false

Mathematical Process

- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☐ Visualization

The senior executives of a local engineering firm are discussing a proposed change in their company's policy. They are reviewing the written arguments. If they find the arguments convincing, they will be prepared to support the change. The burden of proof lies with the authors of the proposal.

Similarly, in mathematics, a **proof** is simply a convincing argument. A proof is neither true nor false. The mathematical statements in the proof are true or false, but the proof itself is simply a logical presentation of the "facts." In your study of mathematics, you will encounter many different kinds of proofs. In this activity, you will explore two broad categories: **direct proof** and **indirect proof**.

In a direct proof, you begin with an assumption; then you work deductively towards a valid conclusion based on that assumption.

The following is an example of a deductive argument (or proof) and two ways you can draw a diagram to show this argument using symbols.

	Argument	**Diagram 1**	**Diagram 2**
Conditional	If it is a cow, then it is a ruminant.	If p, then q	$p \rightarrow q$
Observation	Betsy is a cow.	p	p
Conclusion	Therefore, Betsy is a ruminant.	$\therefore q$	$\therefore q$

Note: The variable *p* represents the hypothesis, and the variable *q* represents the conclusion.

This pattern occurs frequently in direct proofs.

Another frequently occuring pattern involves substitution. An example of this pattern and how it can be represented using symbols follows.

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ☒ Reasoning
- ■ Technology
- ■ Visualization

	Proof	**Diagram**
Conditional	If it is a triangle, then it is a polygon.	$p \rightarrow q$
Observation	*ABC* is a figure formed by joining three noncollinear points.	r
Equivalent Statement	A triangle is the figure formed by joining three noncollinear points.	$r \leftrightarrow p$
Conclusion	Therefore, *ABC* is a polygon.	$\therefore q$

Notice, in the third statement, *triangle* is substituted for *ABC*.

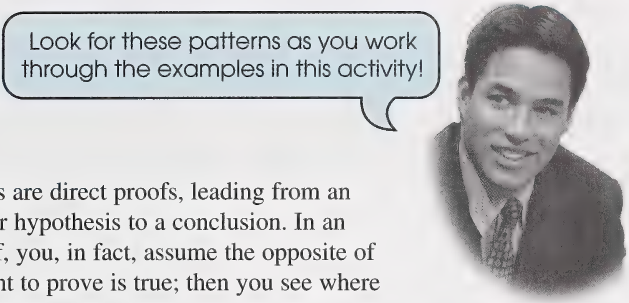
Note: In the preceding proof, the double-headed arrow shows that r and p are equivalent. If statements are equivalent, then one can be substituted for the other.

Another pattern in proofs involves a chain of conditional statements, or implications. An example and diagram follows.

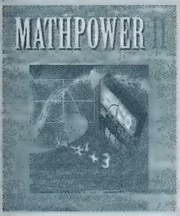
Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ☒ Reasoning
- ■ Technology
- ■ Visualization

	Proof	**Diagram**
Conditional 1	If it is a cow, then it is a ruminant.	$p \rightarrow q$
Conditional 2	If it is a ruminant, then it chews its cud.	$q \rightarrow r$
Observation	Betsy is a cow.	p
Conclusion	Therefore, Betsy chews her cud.	$\therefore r$


Look for these patterns as you work through the examples in this activity!

Not all proofs are direct proofs, leading from an assumption or hypothesis to a conclusion. In an indirect proof, you, in fact, assume the opposite of what you want to prove is true; then you see where this assumption leads you. If it leads to a false statement or contradicts some accepted facts, then the original statement must be true. For example, suppose you want to prove that bison are mammals. To approach the question indirectly, assume the negation is true, that is, bison are not mammals; then investigate where this assumption leads you.



To find out more about indirect proof, turn to page 366 of *MATHPOWER™ 11* and read "Direct and Indirect Proof."

1. Answer the following questions on page 366 of the textbook:

   a. "Explore: Indirect Proof"
   b. questions 1 to 8 of "Inquire"

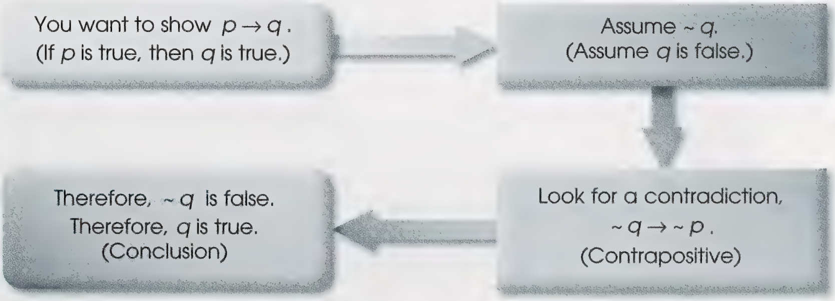
Compare your responses with those in the Appendix, Section 1: Activity 3.

---

proof by contradiction: indirect proof

Indirect proof is also called **proof by contradiction**. This is because you assume the negation of the statement you want to prove and show that it leads to a contradiction. That is, you show that it is impossible for the original statement to be false.

Some indirect proofs may be drawn as follows.


You want to show $p \rightarrow q$.
(If p is true, then q is true.)
Assume $\sim q$.
(Assume q is false.)
Look for a contradiction,
$\sim q \rightarrow \sim p$.
(Contrapositive)
Therefore, $\sim q$ is false.
Therefore, q is true.
(Conclusion)

Recall, from Activity 2, that $q \rightarrow \quad p$ is the contrapositive and that the contrapositive is logically equivalent to the original conditional statement. That is, if $q \rightarrow \quad p$ is true, then $p \rightarrow q$ is true!

Example

Prove that there is no largest prime number.

Solution

Proof by Contradiction

Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☐ Visualization

Assume there is a largest prime number. If there is a largest prime number, then there is a finite number of primes that can be arranged from smallest to largest.

$$\begin{aligned} p_1 &= 2 \\ p_2 &= 3 \\ p_3 &= 5 \\ p_4 &= 7 \\ &\vdots \\ p_n &= \text{largest prime} \end{aligned}$$

Now, the expression $p_1 \times p_2 \times p_3 \times p_4 \times \ldots \times p_n + 1$ is a prime number because dividing this expression by any of the prime numbers from p_1 through to p_n will yield a remainder of 1.

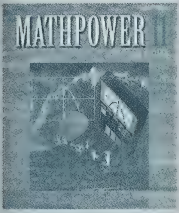
But $p_1 \times p_2 \times p_3 \times p_4 \times \ldots \times p_n + 1$ is larger than p_n. This contradicts the assumption that p_n is the largest prime. Therefore, p_n is not the largest prime, making the original assumption, that there is a largest prime number, false.

Therefore, there is no largest prime.

You will now explore indirect proof in geometry. Don't forget to include any theorems and corollaries in the geometry list you are keeping in your journal.

corollary: an additional statement that follows directly from a theorem

Remember: A theorem is a statement that must be proven before being accepted as true. A corollary is an additional statement that follows directly from a theorem.

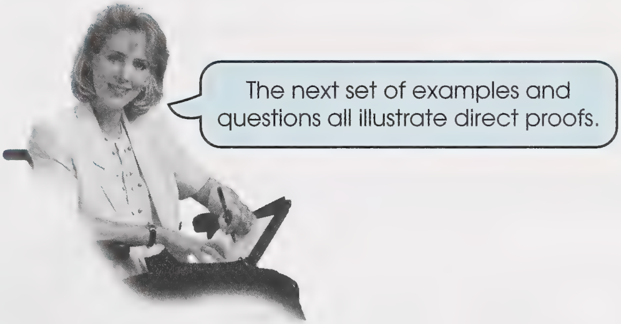


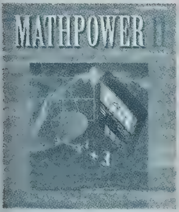
Turn to page 366 of *MATHPOWER™ 11* and read from the red line to Example 1 on page 367; then work through Examples 1, 2, 7, and 8 on pages 367 to 371.

2. Answer the following questions on page 372 of the textbook:

- **a.** questions 1 to 5 of "Practice"
- **b.** questions 12 and 14 "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 1: Activity 3.


The next set of examples and questions all illustrate direct proofs.



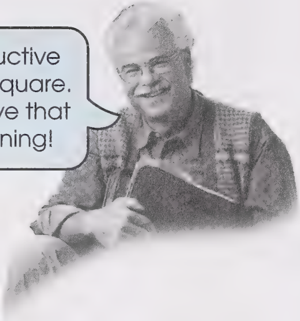
Turn to pages 369 and 370 of *MATHPOWER™ 11* and work through Examples 3 to 6.

3. Answer questions 16, 18, and 19 of "Applications and Problem Solving" on page 372 of the textbook.

Compare your responses with those in the Appendix, Section 1: Activity 3.

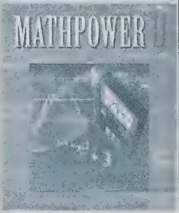
Now Try This

Mathematical Process
- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☐ Visualization



Use logical reasoning and your deductive powers to solve the following magic square. Solving number puzzles is proof positive that you can apply mathematical reasoning!

4. Turn to page 372 of *MATHPOWER™ 11* and answer "NUMBER POWER."



Compare your response with the one in the Appendix, Section 1: Activity 3.

Looking Back

Mathematical Process
- ☒ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

In this activity, you explored some of the elements of direct and indirect proof.

Have you heard the saying, "The proof is in the pudding"? Do some research to analyse the origins and meaning of this saying. Do you agree with it? Write your findings and your thoughts in your journal.

Activity 4: Geometric Proof

Mathematical Process
- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

The illustration on the right side is of Euclid, a fourth-century B.C. mathematician whose work has dominated the study of geometry for over two millenniums. Euclid placed geometry into a logical system of definitions, **axioms** (or **postulates**), and theorems. His greatest work, titled *Elements*—thirteen books on plane and solid geometry—is still used, largely unchanged, by modern mathematicians.

a statement accepted as true without proof

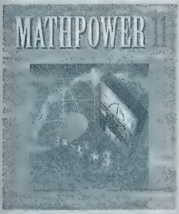
CORBIS/BETTMANN

If you have access to the Internet, you can study Euclid's *Elements*, which has been translated into modern English, at the following website:

http://aleph0.clarku.edu/~djoyce/java/elements/elements.html

The theorems that you will be studying in this module are from Euclidean geometry. No study of Euclidean geometry can begin without a discussion of geometric proof.

In this activity, you will investigate geometric proof. As you work through the examples, keep in mind the characteristics of direct proof and indirect proof, which you explored in Activity 3 of this section. These features are directly applicable to geometric proof.



Turn to page 385 of *MATHPOWER™ 11* and read "Geometric Proofs."

1. Answer the following questions on pages 385 and 386 of the textbook:

a. questions a. to f. of "Explore: Complete the Proof"
b. questions 1 to 3 of "Inquire"

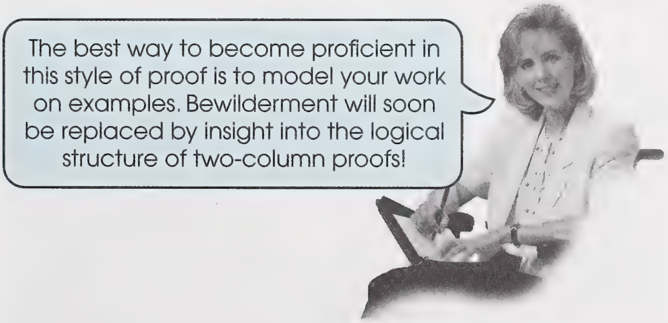
Compare your responses with those in the Appendix, Section 1: Activity 4.

Mathematical Process

☒ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☒ Reasoning
☐ Technology
☐ Visualization

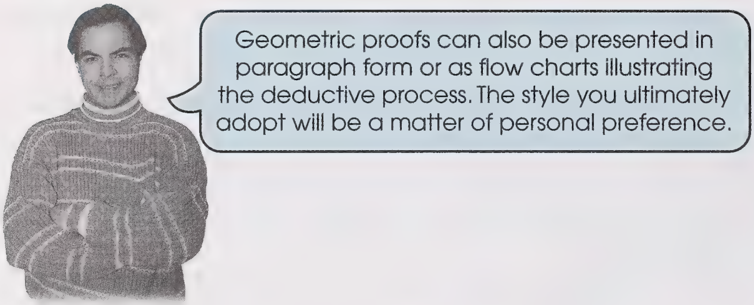
The traditional format for geometric proof consists of two columns. This is the type of proof used in the previous question set.

- The first column consists of statements or assertions linked together logically. The rules of inference, mentioned in Activity 3 of this section, are used to establish the continuous logical flow from assumption to conclusion.
- The second column consists of reasons supporting the statements in the first column. A reason must be given for each statement. The reasons, as pointed out earlier, may be given facts, terms, definitions, axioms (or postulates), number properties, and previously proven theorems.


The best way to become proficient in this style of proof is to model your work on examples. Bewilderment will soon be replaced by insight into the logical structure of two-column proofs!

A property of equality that often appears in geometric proofs is the reflexive property, which simply states that every element is equal to itself. For example, for line segment AB, if you write $AB = AB$, you are applying the reflexive property.

reflexive property: a property of equality that states, "for any a, $a = a$"


Geometric proofs can also be presented in paragraph form or as flow charts illustrating the deductive process. The style you ultimately adopt will be a matter of personal preference.

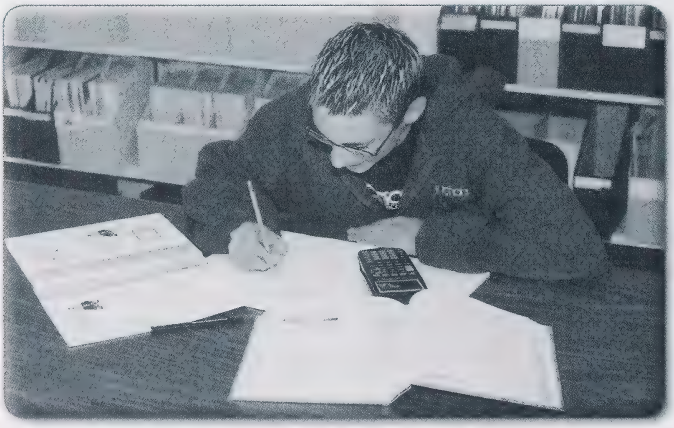
Turn to page 386 of *MATHPOWER™ 11* and read from the red line to the bottom of page 388, working through Examples 1 to 5.

Remember: Add any new theorem and definition to the geometry list in your journal.

2. Answer the following questions on pages 389 to 392 of the textbook:

- **a.** questions 1, 2, 3, 4, and 9 of "Practice"
- **b.** questions 16, 22, and 26 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 1: Activity 4.

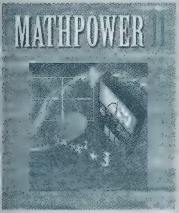


Now Try This

Mathematical Process

- ■ Communication
- ☒ Connection
- ■ Estimation
- ☒ Mental Math
- ☒ Problem Solving
- ■ Reasoning
- ■ Technology
- ☒ Visualization

The ability to interpret diagrams, obtain data from diagrams, and manipulate data mentally are important skills in geometry. In the following set of questions, you will practise these skills with triangles and with parallel lines.



3. Turn to page 383 of *MATHPOWER™ 11* and answer questions 1 to 4 of “Mental Math: Angle Measures.”

Compare your responses with those in the Appendix, Section 1: Activity 4.

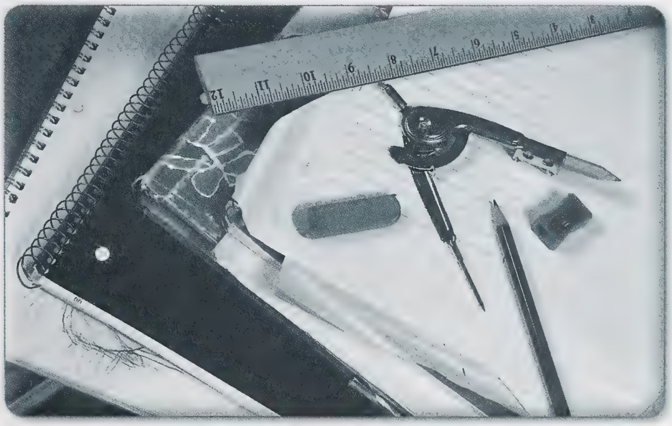
Looking Back

Mathematical Process

- ☒ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ☒ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

In this activity, you were introduced to the elements and structure of geometric proof. These elements and structures will recur in each activity in Section 2. As you work through these activities, you will become increasingly comfortable with using deductive reasoning to prove conjectures.

In your journal, list the elements of mathematical reasoning you would like clarified; then, if possible, exchange lists with another student taking Pure Mathematics 20. Together, discuss your lists and make notes or include examples that will help clarify difficulties.

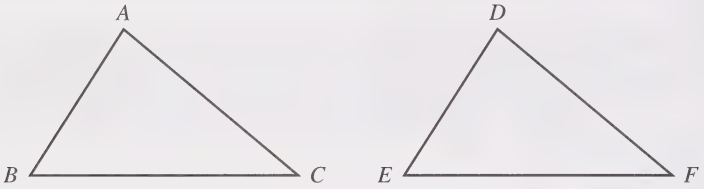


Follow-up Activities

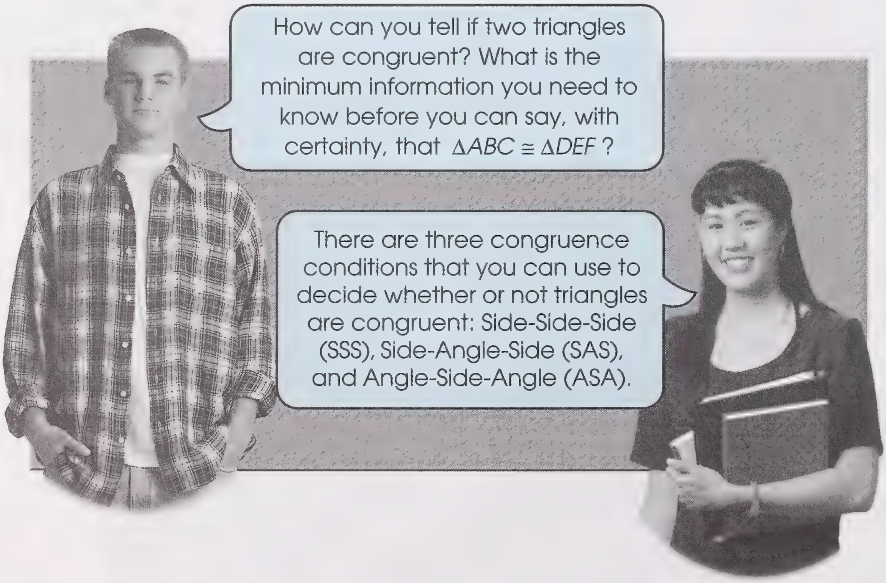
If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

In Activity 4, you were introduced to geometric proof. Many geometric proofs involve, at least in part, a discussion of congruent triangles. Recall that congruent triangles are identical in every respect; their corresponding angles are equal, and their corresponding sides are equal.

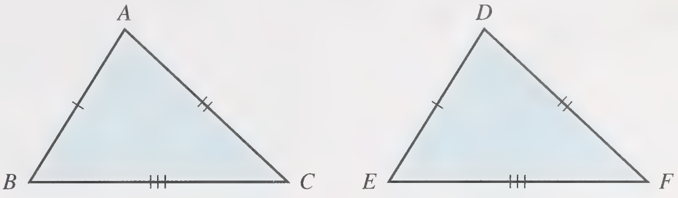


If $\Delta ABC \cong \Delta DEF$, then $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, $AB = DE$, $BC = EF$, and $AC = DF$.

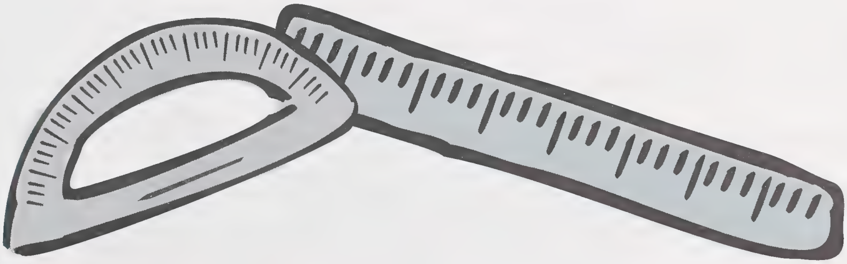

How can you tell if two triangles are congruent? What is the minimum information you need to know before you can say, with certainty, that $\Delta ABC \cong \Delta DEF$?
There are three congruence conditions that you can use to decide whether or not triangles are congruent: Side-Side-Side (SSS), Side-Angle-Side (SAS), and Angle-Side-Angle (ASA).

Side-Side-Side (SSS)

If three sides of one triangle are congruent to the corresponding three sides of a second triangle, then the two triangles are congruent.

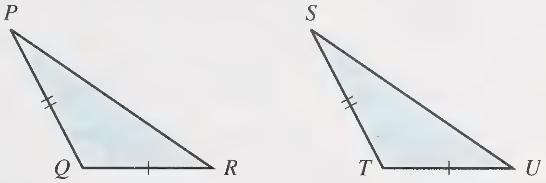


If you knew that $AB = DE$, $AC = DF$, and $BC = EF$, then this is all you need to assert that $\Delta ABC \cong \Delta DEF$.



Side-Angle-Side (SAS)

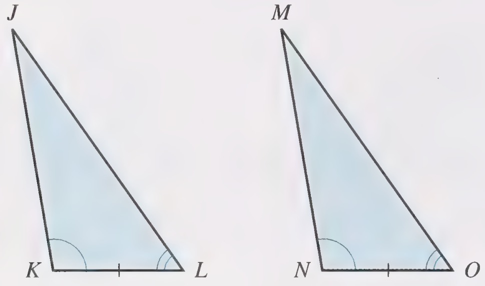
If two sides and the contained angle of one triangle are congruent to the corresponding two sides and contained angle of a second triangle, then the two triangles are congruent.



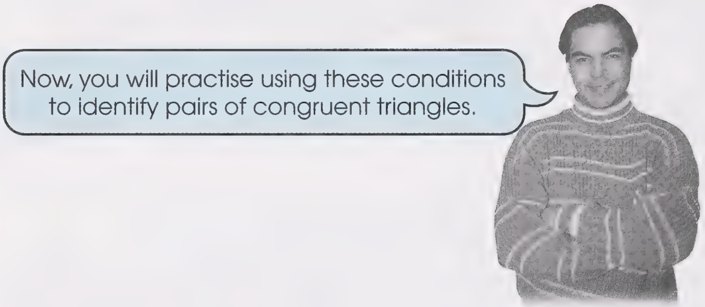
If you knew that $PQ = ST$, $\angle Q = \angle T$, and $QR = TU$, then this is all you need to assert that $\Delta PQR \cong \Delta STU$.

Angle-Side-Angle (ASA)

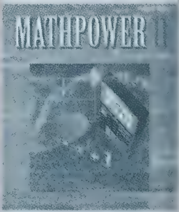
If two angles and the contained side of one triangle are congruent to the corresponding two angles and contained side of a second triangle, then the two triangles are congruent.



If you knew that $\angle K = \angle N$, $KL = NO$, and $\angle L = \angle O$, then this is all you need to assert that $\Delta JKL \cong \Delta MNO$.


Now, you will practise using these conditions to identify pairs of congruent triangles.

Turn to "INVESTIGATING MATH" on page 364 of *MATHPOWER*™ *11* and read "Reviewing Congruent Triangles."



Answer the following questions on pages 364 and 365 of the textbook:

1. questions 1 to 4 of Investigation 1, "Congruent Triangles"

2. questions 1, 3, and 5 of Investigation 2, "Triangles with Common Points and Sides"

Compare your responses with those in the Appendix, Section 1: Follow-up Activities, Extra Help.

Enrichment

In Activity 2, you explored conditional statements. Recall that a conditional statement is a compound statement of the form, "If p, then q." The statement in the *if* clause, p, can be true or false; and the statement in the *then* clause, q, can also be true or false. What combinations of truth values for p and q will yield a true conditional statement?

Example

Discuss the truth value of the following conditional statement Uncle Joe made at a family gathering:

> "If I win the lottery, then I will take all of you to Tashkent."

Solution

The statement in the *if* clause, p, is "Uncle Joe wins the lottery."
The statement in the *then* clause, q, is "He will take us all to Tashkent."

The true and false combinations of these statements appear in the following truth table.

p	q	$p \to q$
T	T	
T	F	
F	T	
F	F	

To complete the table, you must decide which combination yields a true conditional statement and which combination yields a false conditional statement.

Suppose both p and q are true; that is, Uncle Joe wins the lottery and he takes everyone to Tashkent. Uncle Joe told the truth! The conditional statement is true.

Suppose p is true and q is false; that is, Uncle Joe wins the lottery, but he does not take everyone to Tashkent. Uncle Joe did not tell the truth! The conditional statement is false.

Suppose p is false and q is true; that is, Uncle Joe does not win the lottery, but he does take everyone to Tashkent. You can't fault Uncle Joe! He wasn't lying. The conditional statement is true. In logic, a false statement can imply a true statement!

Suppose p is false and q is false; that is, Uncle Joe does not win the lottery, and he does not take everyone to Tashkent. Again, you can't fault Uncle Joe! He did not break his word. The conditional statement is true.

If you summarize these results in the truth table, you can see that the only time a conditional statement is false is when a true *if* clause implies a false *then* clause.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

1. Use the given truth table to decide whether each of the following conditional statements is true or false. Explain.

   a. If $1+1=2$, then $1+2\neq 2$
   b. If $1+1=3$, then $1+2=4$
   c. If $1+1=2$, then $1+2=2$
   d. If $1+1=3$, then $1+2=3$

2. Complete the following truth table to show that a conditional, $p \to q$, and its contrapositive, $\sim q \to \sim p$, are logically equivalent. (That is, they are true at the same times and false at the same times.)

p	q	$p \to q$	$\sim q$	$\sim p$	$\sim q \to \sim q$
T	T				
T	F				
F	T				
F	F				

3. Complete a truth table to show that a conditional, $p \to q$, and its converse, $q \to p$, are not logically equivalent.

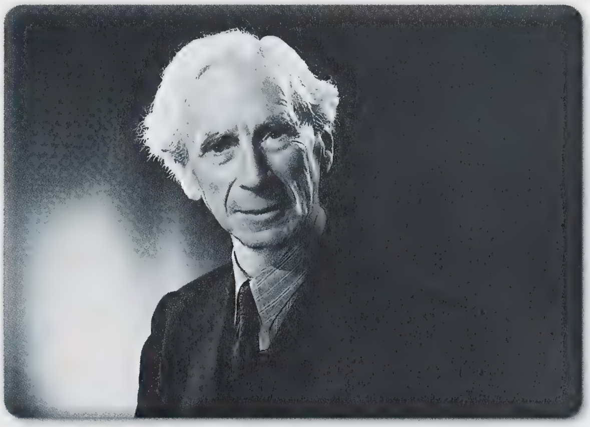
Compare your responses with those in the Appendix, Section 1: Follow-up Activities, Enrichment.

Conclusion

In this section, you explored mathematical reasoning, starting with inductive reasoning and conjecturing. You determined that you can use counterexamples to disprove conjectures, but that you cannot use examples alone to prove conjectures true. To prove mathematical conjectures, you discovered that you must apply deductive reasoning and logic.

As part of your study of logic, you were introduced to both simple mathematical statements and to more-complex, compound statements using the connectives *and*, *or*, and *not*. You also studied the conditional statement and the derived forms—the converse and contrapositive—obtained from the original conditional.

In the last two activities, you applied your newly acquired knowledge of logical structures in mathematical proofs. You explored direct proofs and indirect proofs in both a general context and specifically in geometry.



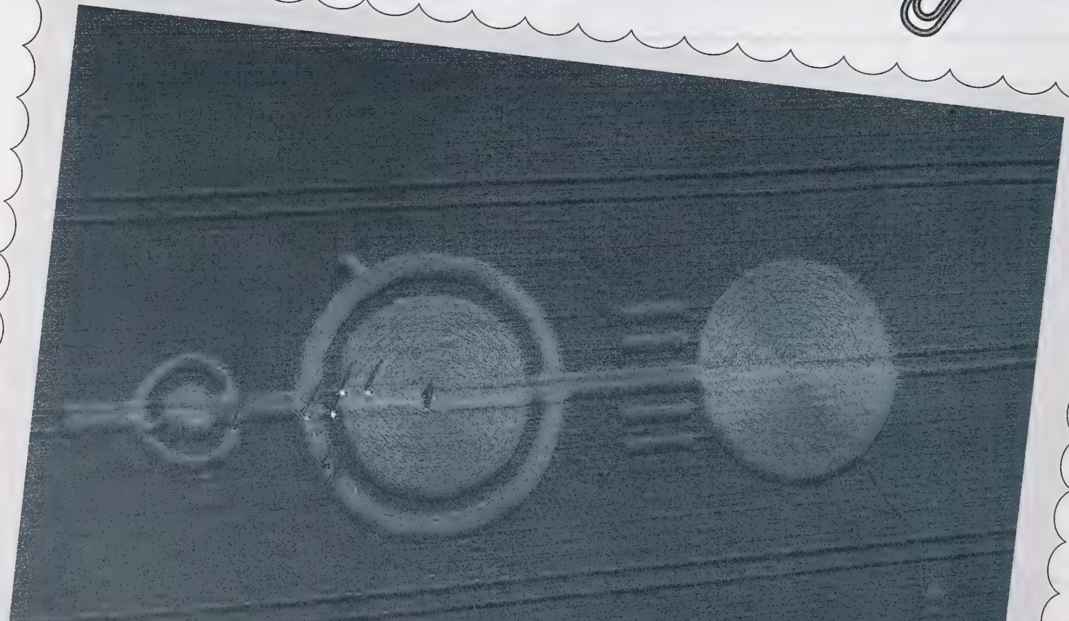
CORBIS/HULTON-DEUTSCH COLLECTION

One of the twentieth century's most eminent logicians was Bertrand Russell. He attempted to demonstrate that all mathematics is built on a sound, logical foundation. Not only was Bertrand Russell a profound mathematical thinker, he did not ignore what he felt were social and political injustices. All his adult life, he was an active advocate for world peace and nuclear disarmament.

Assignment

You are now ready to complete the assignment for Section 1.

Section 2



CORBIS/CHRISTOPHER CORMACK

Circle Geometry

Have you been intrigued by crop circles? Whether they were made by beings not of this world or whether they are elaborate hoaxes, most people are captivated by the intricate designs.

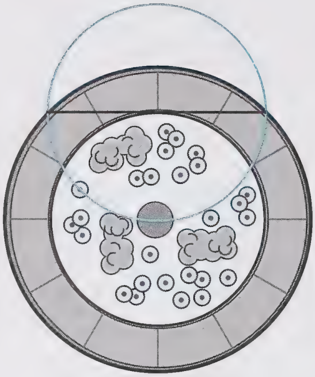
The circle is both utilitarian and aesthetically appealing. Its shape appears everywhere, from wheels and gears to daisies and sophisticated architectural design. The circle is a symbol of perfection and the infinite. Everywhere you look, you will find examples of the circle in nature, in technology, and in design.

In this section, you will explore the geometry of the circle using geometry software, geometric constructions, measurement, and paper folding. You will approach the properties of the circle using both deductive reasoning and inductive reasoning; and, once you have established the properties of a circle, you will apply these properties to solve problems.

Activity 1: Chord Properties

If you have visited a botanical garden, such as the Devonian Botanic Garden near Edmonton, Alberta, you probably noticed that many of the floral displays are surrounded by a circular path for easy viewing. Many formal gardens also incorporate similar arrangements.

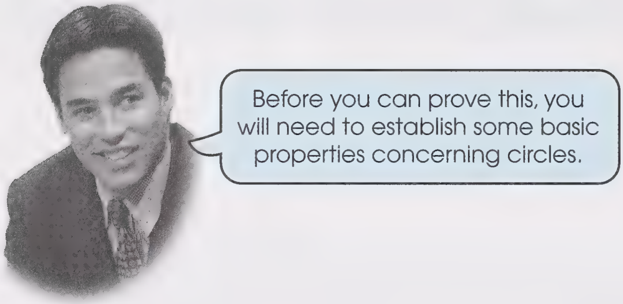
A circular sidewalk around a formal garden is bounded by two concentric circles. The area of this sidewalk can be shown to equal the area of a circle with the endpoints of its diameter on the outer circle and the diameter itself tangent to the inner circle.



Mathematical Process

- ■ Communication
- ☒ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

concentric: having the same centre

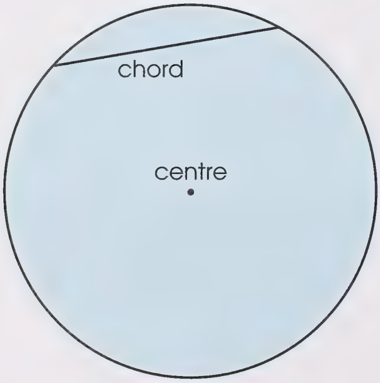

Before you can prove this, you will need to establish some basic properties concerning circles.

Recall that a circle is a set of points equidistant from a fixed point, called the centre. Also, recall that a chord is a line segment that joins two points on the circle. In this activity, you will explore chords and their properties.

circle: a set of points equidistant from a given point

centre of a circle: the fixed point from which all points on the circle are equidistant

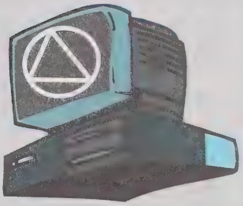
chord: a line segment joining two points on a circle



Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☒ Technology
- ☐ Visualization

If you have access to geometry software, turn to "TECHNOLOGY" on page 394 of *MATHPOWER*™ *11* and read "Exploring Chord Properties with Geometry Software." Then answer all of the questions in Investigations 1 to 3. You may compare your answers with those given for question 1 in the Appendix, Section 2: Activity 1.



If you do not have access to geometry software, turn to "INVESTIGATING MATH" on page 395 of *MATHPOWER*™ *11* and read "Exploring Chord Properties."

1. Answer the following questions on page 395 of the textbook.

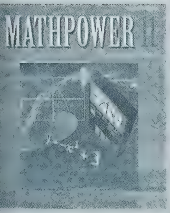
  a. questions 1 to 7 of Investigation 1, "Perpendicular Bisector of a Chord"

  b. questions 1 to 8 of Investigation 2, "Perpendicular from the Centre of a Circle to a Chord"

  c. questions 1 to 9 of Investigation 3, "Line Segment from the Centre of a Circle to the Midpoint of a Chord"

Compare your responses with those in the Appendix, Section 2: Activity 1.

Now, you will apply the chord properties to real-world situations.



Turn to page 396 of *MATHPOWER*™ *11* and read "Chord Properties."

2. Answer the following questions on page 396 of the textbook:

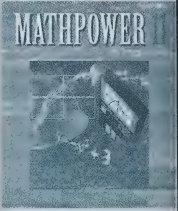
  a. "Explore: Use a Diagram"
  b. questions 1 to 4 of "Inquire"

Compare your responses with those in the Appendix, Section 2: Activity 1.

You have just explored the chord properties inductively, using geometry software or pencil-and-paper techniques. You have also applied these properties to solve real-world problems. Next, you will prove these properties or theorems deductively.

Remember: Include the chord theorems and their corollaries in your geometry list.

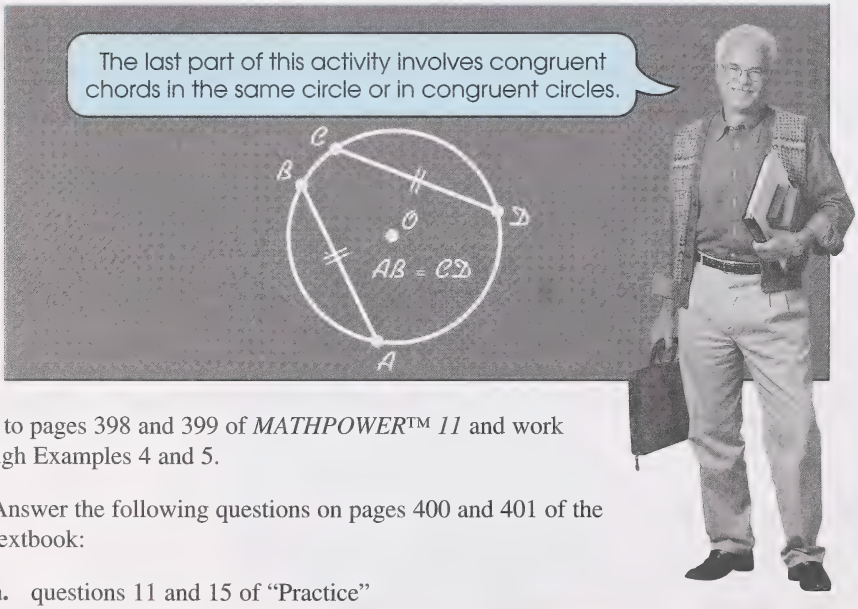
Turn to pages 397 and 398 of *MATHPOWER™ 11* and work through Examples 1 to 3.



3. Answer the following questions on pages 400 and 401 of the textbook:

- **a.** questions 1 to 6, 10, 13, and 14 of "Practice"
- **b.** questions 17, 19, 21, 22, 26, and 30 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 2: Activity 1.


The last part of this activity involves congruent chords in the same circle or in congruent circles.
C
B
D
O
AB = CD
A

Turn to pages 398 and 399 of *MATHPOWER™ 11* and work through Examples 4 and 5.

4. Answer the following questions on pages 400 and 401 of the textbook:

- **a.** questions 11 and 15 of "Practice"
- **b.** questions 24, 32, and 34 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 2: Activity 1.

Mathematical Process

■ Communication
■ Connection
■ Estimation
■ Mental Math
☒ Problem Solving
■ Reasoning
■ Technology
■ Visualization

Now Try This

At the beginning of this activity, you discovered that the area of a sidewalk bounded by two concentric circles is equal to the area of a third circle with a diameter that is tangent to the inner concentric circle and a chord of the outer concentric circle.

5. Use your knowledge of chord properties to prove that the area of the sidewalk equals the area of the third circle.

Compare your responses with those in the Appendix, Section 2: Activity 1.

Mathematical Process

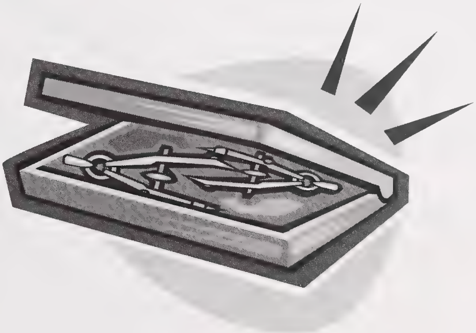
☒ Communication
■ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
■ Technology
■ Visualization

Looking Back

In this activity, you studied the properties of chords in a circle. You should have added the following theorems and corollaries to the geometry list in your journal:

- *The perpendicular bisector of a chord contains the centre of a circle.*
- *The perpendicular bisectors of two non-parallel chords intersect at the centre of the circle.*
- *The perpendicular from the centre of a circle to a chord bisects the chord.*
- *The line segment joining the centre of a circle and the midpoint of a chord is perpendicular to the chord.*
- *Two chords are congruent if and only if they are equidistant from the centre of a circle.*

If possible, compare your geometry list with those of other students taking Pure Mathematics 20. Are there any items that appear on other lists but not on yours? Revise your list, if necessary.



Activity 2: Angle Properties

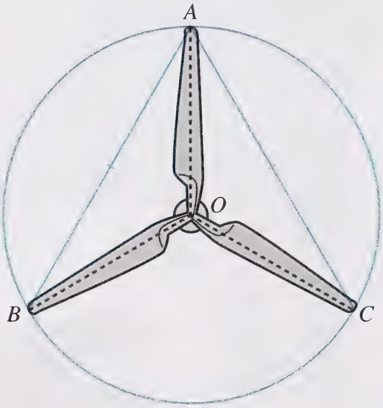
To reduce the reliance on fossil fuels, industry and government are exploring alternative energy sources. Wind farms are a promising possiblity. Giant windmills tap the power of wind to generate electricity, to be fed into an existing power grid, for use in homes and factories. This is a revival of an old idea. In the early part of the twentieth century, many farmsteads on the Canadian Prairies used a system of windmills and storage batteries to light their homes.

Mathematical Process
- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

The blades of the windmills in the photograph are equally spaced. As the blades rotate, their tips trace out a circle.

The measure of the central angle, $\angle BOC$, is

$$\frac{1}{3}\left(360^\circ\right) = 120^\circ$$



What if you measured the angle between the tips of the same two blades at point A? What would be the measure of $\angle BAC$? How would it compare to the measure of $\angle BOC$? In the diagram, $\angle BAC$ is an example of an inscribed angle, an angle with its vertex on the circle and two chords forming its arms.

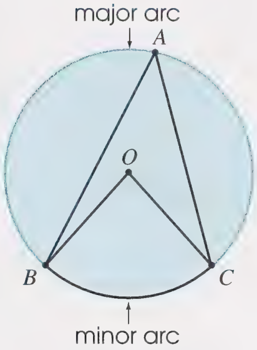
central angle an angle at the centre of a circle subtended by an arc of the circle

inscribed angle an angle that has its vertex on the circle and is subtended by an arc of the circle arc: a continuous piece of the circumference of a circle

minor arc: a smaller arc of a circle defined by two points on the circle

major arc: the larger arc of a circle defined by two points on the circle

There is a relationship between central and inscribed angles. In this case, the arms of both inscribed $\angle BAC$ and central $\angle BOC$ pass through points B and C. Did you know points B and C define two **arcs** (or pieces of the circle)? The smaller piece is called a **minor arc**, and the larger piece is a **major arc**.



$\angle BAC$ and $\angle BOC$ are subtended by minor arc BC.

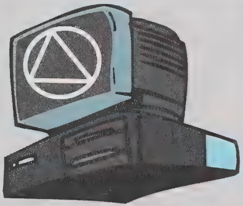
$\angle BAC$ is inscribed (or drawn) in major arc BAC.

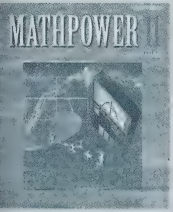
In this activity, you will explore the relationships among central angles, inscribed angles, and the arcs that subtend central and inscribed angles. You will also investigate a special class of quadrilaterals, where all four vertices of these quadrilaterals lie on a circle.

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ■ Visualization

If you have access to geometry software, turn to "TECHNOLOGY" on page 402 of *MATHPOWER™ 11* and read "Exploring Properties of Angles in a Circle Using Geometry Software." Then answer all of the questions in Investigations 1 to 5 on page 402 to 404 of the textbook. You may compare your answers with those given for question 1 in the Appendix, Section 2: Activity 2.





If you don't have access to geometry software, turn to "INVESTIGATING MATH" on page 406 of *MATHPOWER™ 11* and read "Exploring Properties of Angles in a Circle."

1. Answer the following questions on pages 406 and 407 of the textbook:

   a. questions 1 to 7 of Investigation 1, "Central Angles Subtended by Equal Arcs"

   b. questions 1 to 7 of Investigation 2, "Inscribed Angles Subtended by the Same Arc"

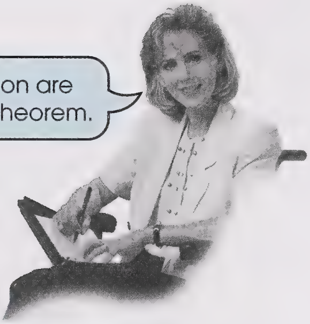
   c. questions 1 to 7 of Investigation 3, "Inscribed Angles Subtended by Equal Arcs"

   d. questions 1 to 7 of Investigation 4, "Angles Inscribed in a Semicircle"

   e. questions 1 to 8 of Investigation 5, "Central and Inscribed Angles Subtended by the Same Arc"

Compare your responses with those in the Appendix, Section 2: Activity 2.

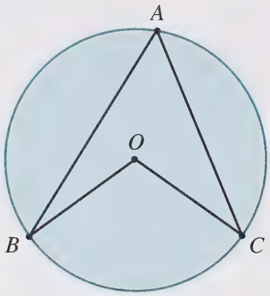
The results of your investigation are summarized by the following theorem.



Angles in a Circle Theorem

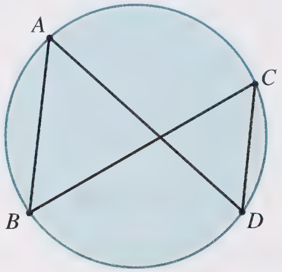
- *The measure of the central angle is equal to twice the measure of the inscribed angle subtended by the same arc.*

In the diagram, $\angle BAC = \frac{1}{2}\angle BOC$ or $\angle BOC = 2\angle BAC$.



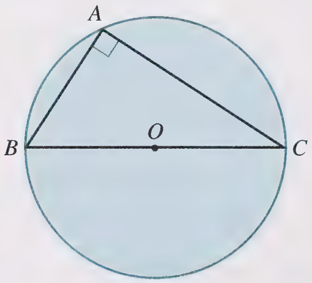
- *Inscribed angles subtended by the same arc or by equal arcs are congruent.*

In the diagram, $\angle BAD = \angle BCD$ and $\angle ABC = \angle ADC$.

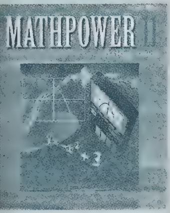


- *The angle inscribed in a semicircle is a right angle.*

In the diagram, $\angle BAC = 90°$.



Now, look at a real-world application.



Turn to page 408 of *MATHPOWER™ 11* and read “Angles in a Circle.”

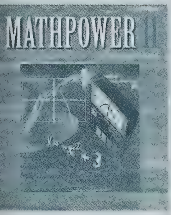
2. Answer the following questions on page 408 of the textbook:

- **a.** questions a. to c. of “Explore: Draw a Diagram”
- **b.** questions 1 and 2 of “Inquire”

Compare your responses with those in the Appendix, Section 2: Activity 2.

You have just explored the angle properties inductively, using geometry software or pencil-and-paper techniques. You have also applied these properties to solve real-world problems. Next, you will prove these properties (or theorems) deductively.

Remember: Include the angle theorems in your geometry list.



Turn to pages 409 to 412 of *MATHPOWER™ 11* and work through Examples 1 to 5.

3. Answer the following questions on pages 412 to 414 of the textbook:

- **a.** questions 1, 3, 5, 6, 8, 9, 10, 13, 14, 19, 20, 22, and 24 of ‘Practice”
- **b.** questions 29, 32.a., 35, and 38 of “Applications and Problem Solving”

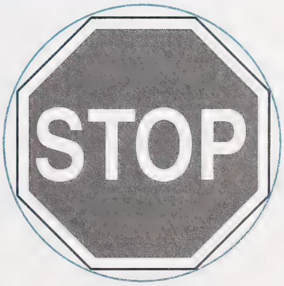
Mathematical Process

■ Communication
☒ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
■ Technology
■ Visualization

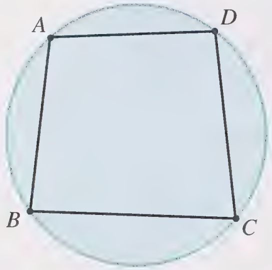
Compare your responses with those in the Appendix, Section 2: Activity 2.

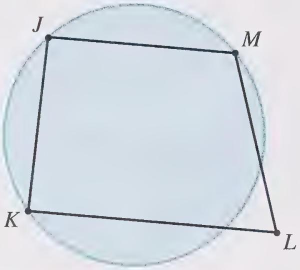
cyclic polygon: a polygon with all of its vertices on the same circle

The shape of a stop sign is an example of a regular polygon—a polygon with equal sides and equal angles. Using the centre of the octagon, you can draw a circle through all eight vertices. Regular polygons are also called **cyclic polygons**. The sides of cyclic polygon are all chords of the same circle.



You can easily draw a quadrilateral in a circle; but not all quadrilaterals are cyclic.

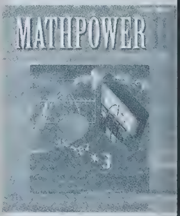




Quadrilateral *ABCD* is cyclic; quadrilateral *JKLM* is not cyclic.

You will now investigate the relationships among the angles of a cyclic quadrilateral and the conditions necessary to decide whether a given quadrilateral is cyclic.

Turn to page 415 of *MATHPOWER*™ *11* and read "Cyclic Quadrilaterals."



4. Answer the following questions on page 415 of the textbook:

a. questions a. and b. of "Explore: Use a Diagram"
b. questions 1 to 7 of "Inquire"

Compare your responses with those in the Appendix, Section 2: Activity 2.

In the preceding questions, you investigated the following theorem:

Theorem

The opposite angles of a cyclic quadrilateral are supplementary.

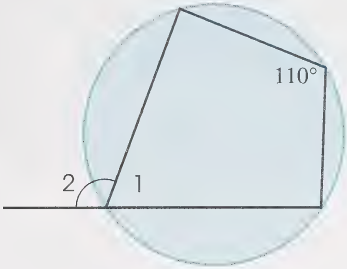
There is a corollary that follows immediately from this theorem.

Corollary

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Example

Find the values of the numbered angles in the diagram. Give reasons for your answers.



Solution

Because the quadrilateral is cyclic, the opposite angles are supplementary.

$$\therefore\ \angle 1 + 110° = 180°$$
$$\angle 1 = 70°$$

Now, $\angle 2$ can be found two different ways.

Method 1

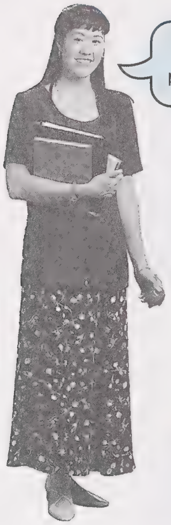
Because $\angle 1$ and $\angle 2$ are supplementary,

$$\angle 1 + \angle 2 = 180°$$
$$70° + \angle 2 = 180°$$
$$\angle 2 = 110°$$

Method 2

Because $\angle 2$ is an exterior angle of a cyclic quadrilateral, it is equal to the interior opposite angle.

$$\therefore\ \angle 2 = 110°$$


You will now apply these results in problem situations.

Turn to page 416 of *MATHPOWER*™ *11* and work through Examples 1 and 2.

5. Answer the following questions on pages 419 and 420 of the textbook:

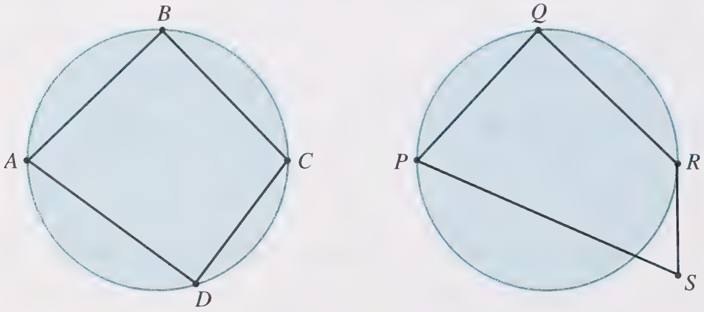
a. questions 1 and 3 to 8 of "Practice"
b. questions 15 and 16 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 2: Activity 2.

You will now consider conditions necessary to assert that four points define a cyclic quadrilateral; that is, all four points lie on the same circle. In the previous set of questions, you applied the following theorem:

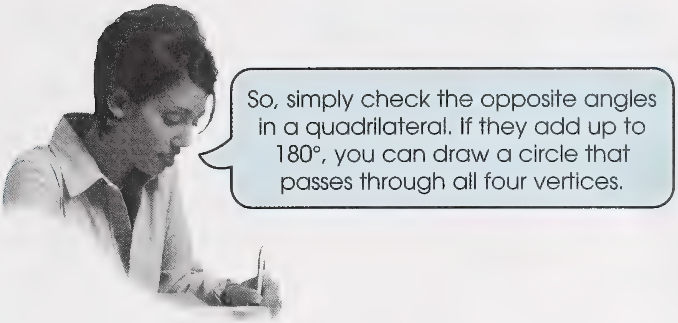
Theorem

The opposite angles of a cyclic quadrilateral are supplementary.



If *ABCD* is a cyclic quadrilateral, then $\angle A + \angle C = 180^\circ$ and $\angle B + \angle D = 180^\circ$. If *PQRS* is not a cyclic quadrilateral, then $\angle P + \angle R \neq 180^\circ$ and $\angle Q + \angle S \neq 180^\circ$.

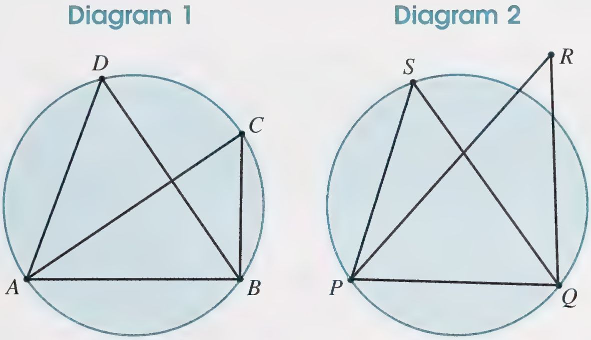
Also, the converses are true. If $\angle A + \angle C = 180^\circ$ and $\angle B + \angle C = 180^\circ$, then *ABCD* is a cyclic quadilateral. Also, if $\angle P + \angle R \neq 180^\circ$ and $\angle Q + \angle S \neq 180^\circ$, then *PQRS* is not a cyclic quadrilateral.


So, simply check the opposite angles in a quadrilateral. If they add up to 180°, you can draw a circle that passes through all four vertices.

Theorem

If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

There is another way of determining whether four points define a cyclic quadrilateral.

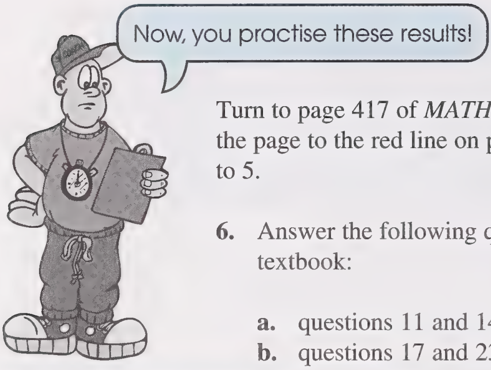


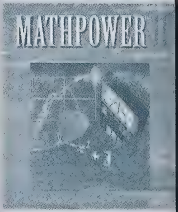
Recall from the first part of this activity that if, in a circle, two inscribed angles are subtended by the same arc, then these angles are equal in measure. Therefore, in Diagram 1, $\angle C = \angle D$. The converse is also true.

If $\angle C = \angle D$, then $ABCD$ is a cyclic quadrilateral. In Diagram 2, $PQRS$ is not a cyclic quadrilateral. Therefore, $\angle R \neq \angle S$. Again, the converse is true. If $\angle R$ and $\angle S$ are not equal in measure, you can't draw the same circle through P, Q, R, and S.

Theorem

If the line segment joining two points subtends two equal angles at two other points on the same side of the segment, then the four points are concyclic.


Now, you practise these results!



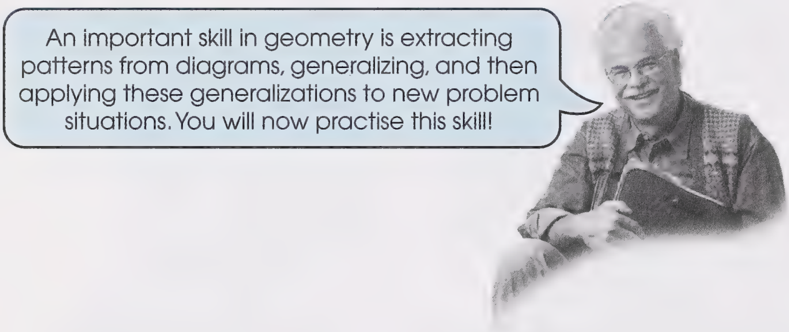
Turn to page 417 of *MATHPOWER™ 11* and read from the top of the page to the red line on page 419, working through Examples 3 to 5.

6. Answer the following questions on pages 420 and 421 of the textbook:

- **a.** questions 11 and 14 of "Practice"
- **b.** questions 17 and 23 of "Applications and Problem Solving"

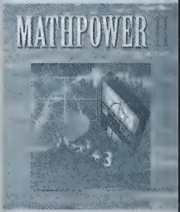
Compare your responses with those in the Appendix, Section 2: Activity 2.

Now Try This


An important skill in geometry is extracting patterns from diagrams, generalizing, and then applying these generalizations to new problem situations. You will now practise this skill!

Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☒ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☒ Visualization



7. Turn to page 421 of *MATHPOWER™ 11* and answer questions 1 and 2 of "PATTERN POWER"

Compare your responses with those in the Appendix, Section 2: Activity 2.

Looking Back

In this activity, you explored the following theorems dealing with angle properties in a circle:

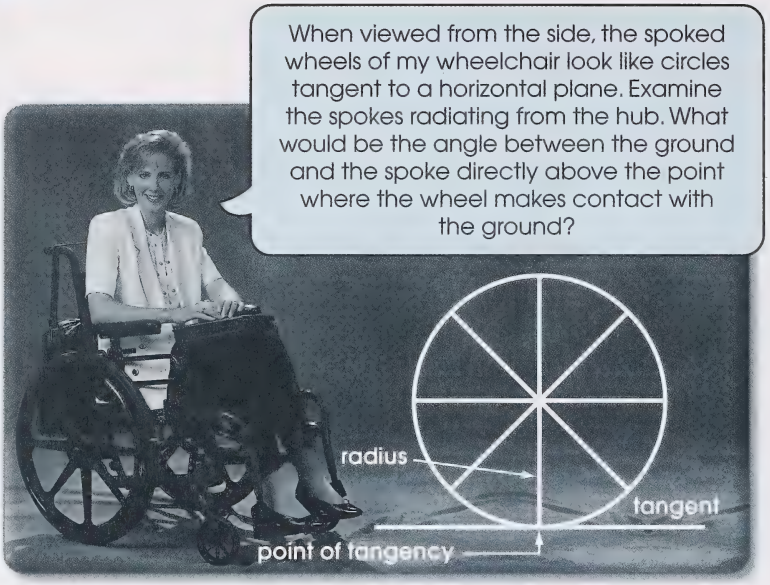
- *The measure of the central angle is equal to twice the measure of the inscribed angle subtended by the same arc.*
- *Inscribed angles subtended by the same arc, or by equal arcs, are congruent.*
- *The angle inscribed in a semicircle is a right angle.*
- *The opposite angles of a cyclic quadrilateral are supplementary.*
- *The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.*
- *The line segment joining two vertices of a cyclic quadrilateral subtends equal angles at the other two vertices on the same side of the segment.*
- *If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.*
- *If an exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is cyclic.*
- *If the line segment joining two points subtends two equal angles at the two other points on the same side of the segment, then the four points are concyclic.*

Read over these theorems; then, in your journal, put them into your own words and illustrate each using a diagram. Write the theorems that are paired with converses as *if and only if* statements. Compare your work with the work of another student taking Pure Mathematics 20.

Mathematical Process

☒ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☒ Reasoning
☐ Technology
☐ Visualization

Activity 3: Tangents to a Circle


When viewed from the side, the spoked wheels of my wheelchair look like circles tangent to a horizontal plane. Examine the spokes radiating from the hub. What would be the angle between the ground and the spoke directly above the point where the wheel makes contact with the ground?
radius
tangent
point of tangency

Mathematical Process

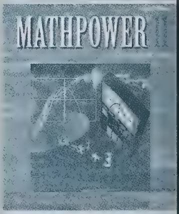
- ☐ Communication
- ☒ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☒ Visualization

Do you agree that a **tangent** drawn to a circle is perpendicular to the radius at the **point of tangency**? What about the converse? If a line is perpendicular to a radius at its outer endpoint, then the line is tangent to the circle. In fact, these are two of the theorems involving circles and tangents that you will explore in this activity.

tangent: a line that touches a curve at a single point

point of tangency: the point where a tangent touches a curve

Turn to page 426 of *MATHPOWER*™ *11* and read "Tangents to a Circle."

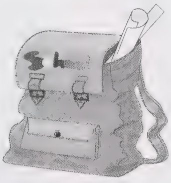


1. Answer the following questions on pages 426 and 427 of the textbook:

   a. questions a. to c. of Investigation 1 of "Explore: Use the Diagrams"

   b. questions a. to c. of Investigation 2 of "Explore: Use the Diagrams"

   c. questions 1 to 3 of "Inquire"



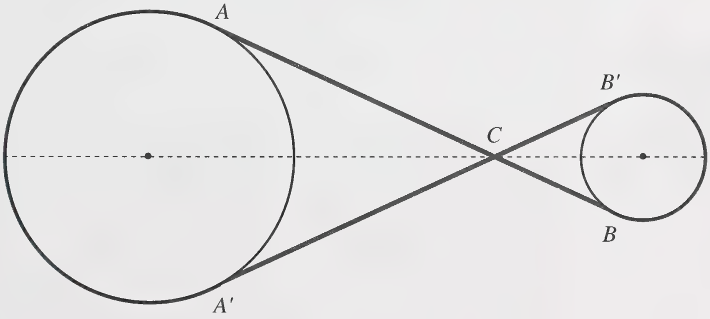
Compare your responses with those in the Appendix, Section 2: Activity 3.

CORBIS

Mathematical Process

☐ Communication
☒ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☐ Reasoning
☐ Technology
☐ Visualization

Early in the twentieth century, prairie farmers relied on threshing crews during harvest time. The threshing machine was powered by a tractor using a belt-and-pulley system. The belt ran from the tractor to the threshing machine. When the farmers needed to reverse the direction of rotation, they simply put a twist in the belt, as shown in the following diagram.



Mathematical Process

☐ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☒ Reasoning
☐ Technology
☒ Visualization

In the diagram, how do the lengths of the tangent segments drawn from point C compare? Look at the dotted line drawn through the centres of the circles. The figure is symmetrical about this line. This means if you were to fold the diagram about this line, points A and A' would coincide and points B and B' would coincide, making $CA = CA'$ and $CB = CB'$.

Theorem

The tangent segments to a circle from any external point are congruent.

Now, consider a few more examples, and then practise your skills.

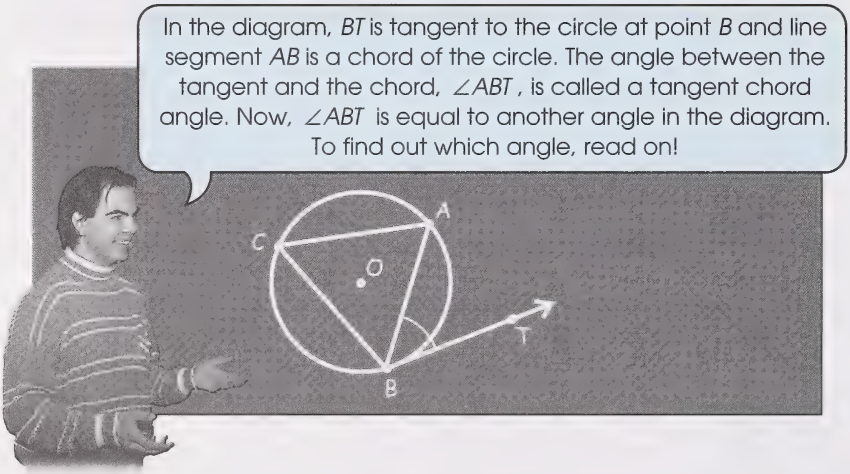
Turn to pages 427 to 429 of *MATHPOWER™ 11* and work through Examples 1 to 5.

2. Answer the following questions on pages 431 and 432 of the textbook:

a. questions 1 to 6 of "Practice"
b. questions 15, 18, 19, 20, 23, 27, and 31 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 2: Activity 3.

The last part of this activity deals with the angles between tangents and chords.


In the diagram, *BT* is tangent to the circle at point *B* and line segment *AB* is a chord of the circle. The angle between the tangent and the chord, $\angle ABT$, is called a tangent chord angle. Now, $\angle ABT$ is equal to another angle in the diagram. To find out which angle, read on!
C
A
O
T
B

Turn to page 430 of *MATHPOWER™ 11* and work through Examples 6 and 7.

3. Answer the following questions on pages 431 and 432 of the textbook:

a. questions 7, 9, 11, and 13 of "Practice"
b. questions 16 and 26 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 2: Activity 3.

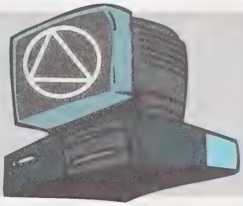
Now Try This

The following question set should reinforce the Tangent Chord Theorem. Recall that this theorem states, “The angle between a tangent and a chord is equal to the inscribed angle on the opposite side of the chord.”

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ■ Visualization

If you have access to geometry software, turn to page 423 of *MATHPOWER™ 11* and answer questions 1 to 8 of Investigation 4, “Angle Between a Tangent and a Chord.” You may compare your answers with those given for question 4 in the Appendix, Section 2: Activity 3.



4. If you do not have access to geometry software, turn to page 425 of *MATHPOWER™ 11* and answer questions 1 to 8 of Investigation 4, “Angle Between a Tangent and a Chord.”

Compare your responses with those in the Appendix, Section 2: Activity 3.

Looking Back

In this activity, you explored the following theorems that involve tangents to circles:

- *A tangent to a circle is perpendicular to the radius at the point of tangency.*
- *If a line is perpendicular to a radius at its endpoint, then the line is tangent to the circle.*
- *The tangent segments to a circle from any external point are congruent.*

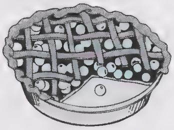
Mathematical Process

- ☒ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

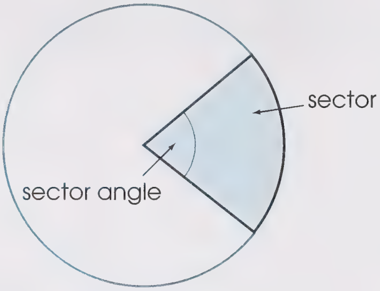
Have you heard the phrase, “go off on a tangent”? What does this mean? Is it related in any way to circles and tangents? Use your research skills to answer these questions, and record your findings in your journal.

Activity 4: Arc Lengths, Sector Angles, and Polygons and Their Angles

Suppose you were to share a pie with four other people so that everyone would receive the same amount. If you cut your slice from the pie first, you would remove $\frac{1}{5}$ of the pie, or 20%. Would you have removed 20% of the outer crust (circumference)? What about the top crust (the area of the circle)?



In a circle, a wedge-shaped region, like a piece of pie, bounded by two radii and the arc between them is called a **sector** of a circle. The central angle of a sector may also be called the **sector angle**.



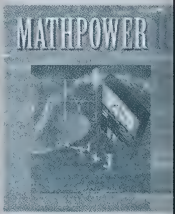
In this activity, you will explore the relationship between sector angle and arc length and sector area. In addition, you will investigate polygons and their angles.

Mathematical Process

- ■ Communication
- ☒ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

sector: a region bounded by two radii and an arc of the circle

sector angle: the central angle formed by a sector of a circle



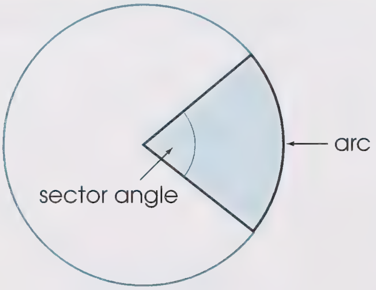
Turn to page 433 of *MATHPOWER™ 11* and read "Arc Length and Sector Area."

1. Answer the following questions on page 433 of the textbook:

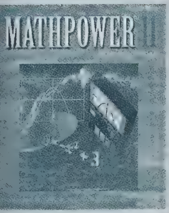
a. questions a. to d. of "Explore: Interpret the Information"
b. questions 1 to 5 of "Inquire"

Compare your responses with those in the Appendix, Section 2: Activity 4.

In the previous exploration, you discovered that the length of the arc and the sector area, for a sector angle θ, are fractions of 360°. This discovery allowed you to then develop a formula for calculating the area, A, and the arc length, ℓ, of a sector. These formulas are as follows:



$$A = \frac{\theta}{360^\circ} \times \pi r^2 \qquad \ell = \frac{\theta}{360^\circ} \times 2\pi r$$



Turn to page 434 and 435 of *MATHPOWER™ 11* and work through Examples 1 and 2.

2. Answer the following questions on pages 435 to 437 of the textbook:

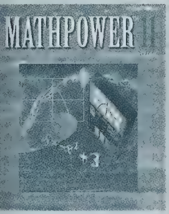
- **a.** questions 2, 7, 9, 15, and 27 of "Practice"
- **b.** questions 34, 36, and 42 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 2: Activity 4.

---

olygon: a closed
igure with three or
ore sides

Next, you will investigate angles and **polygons**.



Turn to page 439 of *MATHPOWER™ 11* and read "Angles and Polygons."

3. Answer the following questions on pages 439 and 440 of the textbook:

- **a.** "Explore: Look for a Pattern"
- **b.** questions 1 to 4 of "Inquire"

Compare your responses with those in the Appendix, Section 2: Activity 4.

---

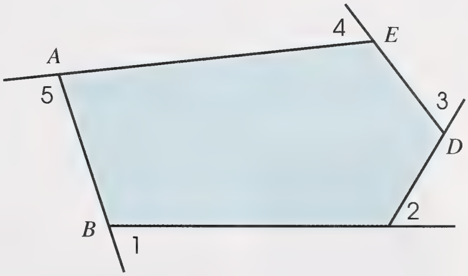
onvex polygon: a
olygon in which
egments joining any
wo points in the
olygon lie entirely
ithin the polygon

One way of analysing the sum of the interior angles of a **convex polygon** is to look at the exterior angles first.

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ☒ Mental Math
- ☒ Problem Solving
- ☒ Reasoning
- ■ Technology
- ☒ Visualization

Consider the following convex polygon.



As you move counterclockwise around the polygon, starting at A and going along AB, you must turn through exterior angle 1, then through angle 2, and so on. To complete your journey, and be pointed in the same direction as when you started, you will turn through a complete circle, or 360°.

In this case, $\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5 = 360°$.

This relationship holds for a polygon with any number of sides.

Theorem

The sum of the measures of the exterior angles of any convex polygon is 360°.

Look at the diagram once again. Exterior $\angle 1$ and interior $\angle B$ are supplementary; therefore, $\angle 1 + \angle B = 180°$. This is true for all such pairs of angles.

Now, for a polygon with n sides, there would be n pairs of exterior and interior angles. Therefore, the sum of the measures of all the exterior angles and interior angles is $180° n$.

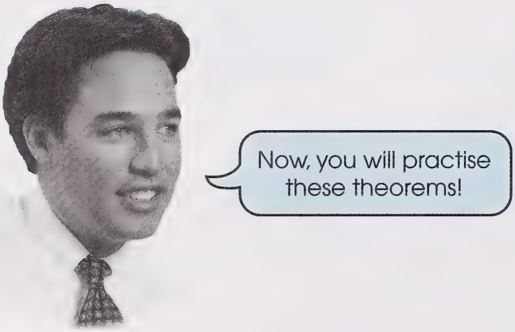
$$(\text{sum of the interior angles}) + (\text{sum of the exterior angles}) = 180° n$$

But the sum of the exterior angles is 360°.

$$\begin{aligned} \therefore\ (\text{sum of the interior angles}) + 360° &= 180° n \\ &= 180° n - 360° \\ &= 180°(n-2) \end{aligned}$$

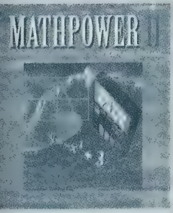
Theorem

The sum of the interior angles of any polygon with n sides is $180°(n-2)$.


Now, you will practise these theorems!

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ☒ Reasoning
- ■ Technology
- ☒ Visualization



Turn to page 440 of *MATHPOWER™ 11* and read from the red line to the red line on page 442, working through Examples 1 to 3.

4. Answer the following questions on pages 442 and 443 of the textbook:

a. questions 1, 3, 5, 11, 14, 20, and 26 of "Practice"
b. questions 33, 34.a., and 35 of "Applications and Problem Solving"

Compare your responses with those in the Appendix, Section 2: Activity 4.

Now Try This

5. The Exterior Angle Sum Theorem states, "For any convex polygon, the sum of the exterior angles is 360°." Do the exterior angles of a concave polygon always add up to 360°? Support your answer with a diagram.

Mathematical Process

☐ Communication
☐ Connection
☒ Estimation
☐ Mental Math
☐ Problem Solving
☒ Reasoning
☐ Technology
☒ Visualization

Compare your responses with those in the Appendix, Section 2: Activity 4.

Looking Back

Mathematical Process

☒ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☐ Reasoning
☐ Technology
☐ Visualization

In this activity, you explored and applied the following theorems:

- *The sum of the interior angles of any polygon with n sides is* $180^\circ(n-2)$.
- *The measure of each interior angle of a regular polygon with n sides is*
  $$\frac{180^\circ(n-2)}{n}$$
- *The sum of the exterior angles of any convex polygon is* 360°.
- *The measure of each exterior angle of a regular polygon with n sides is*
  $$\frac{360^\circ}{n}$$

Have you added these theorems to the geometry list in your journal? Make sure your list is complete. Carefully go through the activities in this section and check your list.

Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

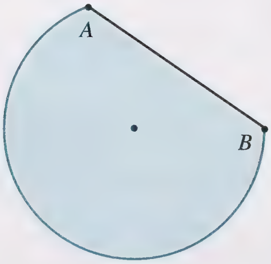
Most of the theorems in circle geometry can be demonstrated by paper folding. Begin by examining the chord theorems.

Mathematical Process

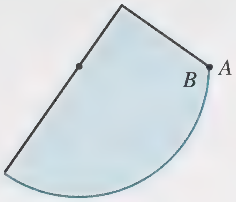
- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☒ Visualization

Step 1: Use a compass to draw a circle on a blank piece of paper. Put a dot on the centre. Then carefully cut out the circle.

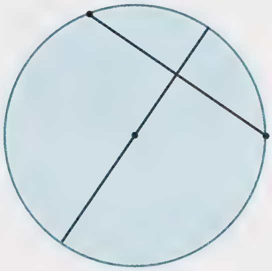
Step 2: Mark two points, *A* and *B*, on the circumference; then fold the circle along chord *AB* (as shown).



Step 3: To obtain the perpendicular bisector of chord AB, simply fold the chord in half so that point A coincides with B (as shown).



Notice that the circle is divided in half, the arcs coincide, and the crease passes through the centre. Now unfold the circle. You should see the following.



Next, answer the following questions based on your circle.

1. Why is the crease through the centre of the circle the perpendicular bisector of the chord?
2. What chord theorems are illustrated by the creases on your circle?
3. Suppose you forgot to mark the centre of the circle. How could you locate the centre through paper folding?

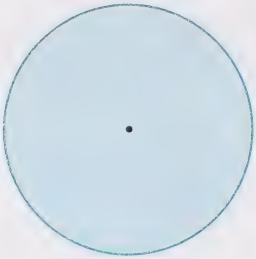
Compare your responses with those in the Appendix, Section 2: Follow-up Activities, Extra Help.

Next, you will use paper folding to show that an angle inscribed in a semicircle is a right angle.

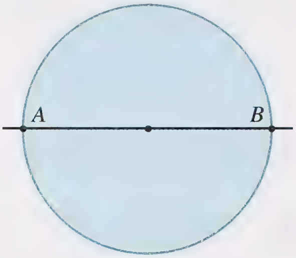
Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☒ Visualization

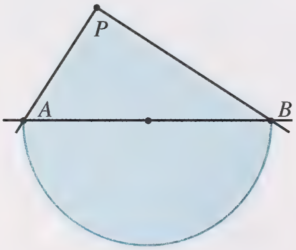
Step 1: Draw a circle on a blank sheet of paper, and mark the centre.



Step 2: Fold the sheet so that the crease passes through the centre. (The fold divides your circle into two semicircles.) Mark the ends of the diameter with *A* and *B* (as shown).



Step 3: Mark point *P* anywhere on one of the semicircles. Fold the sheet of paper along *AP* and then along *BP*.



4. How can you tell from your folded sheet that $\angle APB$ is a right angle?

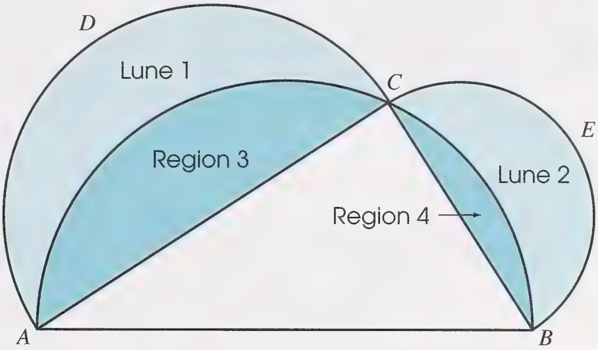
5. How would you fold the paper in order to obtain a crease that is tangent to a point on the circle?

Compare your responses with those in the Appendix, Section 2: Follow-up Activities, Extra Help.

Enrichment

Hippocrates of Chios, born 360 B.C., was a skilled geometer. One of the geometric problems he explored was squaring the circle, using only a compass and a straightedge, trying to construct a square equal in area to a given circle. This construction has been shown by modern mathematicians to be impossible. However, in his attempt, he discovered a relationship between lunes—crescent-shaped areas bounded by arcs of circles.

In the diagram on the right, Lune 1 is bounded by semicircle *ADC* and an arc of the semicircle *ACB*. Lune 2 is bounded by semicircle *BEC* and an arc of the semicircle *ACB*. Hippocrates discovered that the sum of the areas of the two lunes is equal to the area of ΔABC.



Proof

Identify the regions between the lunes and the sides of the triangle as 3 and 4 (as shown).

$$\begin{aligned}\text{Area of Lune } 1 &= \text{Area of semicircle } ADC - \text{Region } 3\\ &= \frac{1}{2}\left[\pi\left(\frac{AC}{2}\right)^2\right] - \text{Region } 3\\ &= \frac{1}{8}\pi(AC)^2 - \text{Region } 3\end{aligned}$$

$$\begin{aligned}\text{Area of Lune 2} &= \text{Area of semicircle } BEC - \text{Region 4}\\ &= \frac{1}{2}\left[\pi\left(\frac{BC}{2}\right)^2\right] - \text{Region 4}\\ &= \frac{1}{8}\pi(BC)^2 - \text{Region 4}\end{aligned}$$

$$\begin{aligned}\therefore \text{ Area of Lune 1} + \text{Area of Lune 2} &= \frac{1}{8}\pi(AC)^2 + \frac{1}{8}\pi(BC)^2 - \text{Region 3} - \text{Region 4}\\ &= \frac{1}{8}\pi\left[(AC)^2 + (BC)^2\right] - (\text{Region 3} + \text{Region 4})\end{aligned}$$

Because arc ACB is a semicircle, ΔABC is a right triangle.

$$\therefore (AC)^2 + (BC)^2 = (AB)^2$$

$$\begin{aligned}\therefore \text{ Area of Lune 1} + \text{Area of Lune 2} &= \frac{1}{8}\pi(AB)^2 - (\text{Region 3} + \text{Region 4})\\ &= \frac{1}{2}\pi\left[\left(\frac{AB}{2}\right)^2\right] - (\text{Region 3} + \text{Region 4})\\ &= \text{Area of semicircle } ACB - (\text{Region 3} + \text{Region 4})\\ &= \Delta ABC\end{aligned}$$

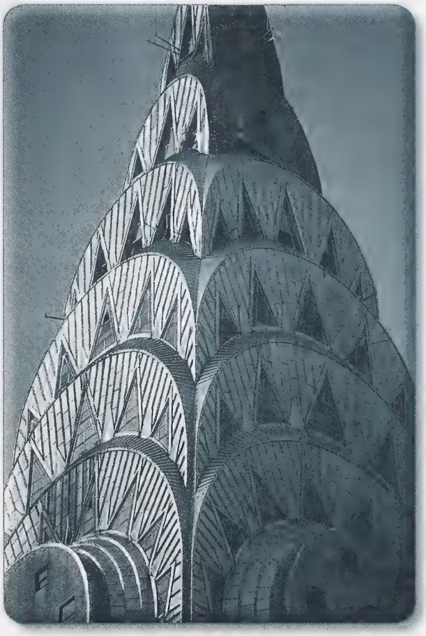
Now try the following questions.

1. What would be the total area of the two lunes constructed on a 3-4-5 right triangle?

2. What would be the maximum area of the sum of the lunes for a triangle inscribed in a semicircle of diameter 5?

Compare your responses with those in the Appendix, Section 2: Follow-up Activities, Enrichment.

Conclusion


Chrysler Building, New York

In this section, you explored the geometry of the circle. In your exploration, you used both deductive reasoning and inductive reasoning. With the help of geometry software, geometric constructions and measurement, and paper folding, you investigated chord, angle, and tangent properties. Once you established the properties of circles, you applied these properties to solve problems. In addition to geometry of the circle, you studied polygons and their angles.

Circles and polygons are familiar shapes in modern architecture. Buildings that are more than box-shaped appeal to the eye, making a city landscape more interesting. The Chrysler Building in the city of New York, for example, exhibits some intricate art-deco-style details and displays how the designer, William van Alen, used curves and angles to captivate the attention of the viewer. Built in 1930 and considered by many to be the most beautiful building in New York, at 77 storeys it was once the tallest building in the city.

Of course, the circle has much more than a simple aesthetic appeal; it has countless applications, from the wheel to the moving parts of sophisticated modern technology. Is it any wonder that the circle plays such a prominent role in the study of geometry?

Assignment

You are now ready to complete the assignment for Section 2.

Module Summary

In Section 1, you explored mathematical reasoning. You investigated mathematical conjectures arising through inductive reasoning and discovered that you could reject a false conjecture quickly by citing a single counterexample. To prove a conjecture true was a much more complicated process involving logic. Finally, you studied direct proof and indirect proof and how they could be adapted to geometry.

In Section 2, you explored circle geometry. You investigated chord properties, angle relationships, and tangent theorems. You then proved these properties and extended your exploration to polygons and their angles.

The exploration of shape and space is an integral part of the human intellectual journey. From the empirical approach of the Egyptians and Babylonians to the logical foundations of the Greek geometers (like Aristotle, Pythagoras, and Euclid), mathematicians have been captivated by geometric form and mathematical structure. This journey continues with modern mathematicians exploring non-Euclidian geometries and broader problems, such as how all mathematics can be built on a logical foundation.

Final Module Assignment

You are now ready to complete the Final Module Assignment.

Appendix

Glossary
Suggested Answers

Glossary

acute triangle: a triangle with three acute angles

arc: a continuous piece of the circumference of a circle

axiom: a statement accepted as true without proof

biconditional statement: a statement formed by combining a conditional and its converse, using *if and only if*

central angle: an angle at the centre of a circle subtended by an arc of the circle

centre of a circle: the fixed point from which all points on the circle are equidistant

chord: a line segment joining two points on a circle

circle: a set of points equidistant from a given point

compound statement: two statements linked together with a connective, such as *and*, *or*, or *not*

concentric: having the same centre

conclusion: the part in the *then* clause of a conditional statement

conditional statement: an *if...then* statement

conjecture: a generalization made through inductive reasoning

contrapositive: a conditional statement formed by negating both the *if* and *then* clauses of a converse

converse: a conditional statement formed by interchanging the *if* and *then* clauses of another conditional statement

convex polygon: a polygon in which segments joining any two points in the polygon lie entirely within the polygon

corollary: an additional statement that follows directly from a theorem

counterexample: an example that disproves a conjecture

cyclic polygon: a polygon with all of its vertices on the same circle

deductive reasoning: a process of demonstrating that, if certain statements are true, then other statements follow logically from them

direct proof: a proof in which you begin with an assumption and work deductively towards a valid conclusion

hypothesis: the part in the *if* clause of a conditional statement

implication: a conditional statement

indirect proof: a process of arguing in which you assume the conclusion you wish to demonstrate is false

inductive reasoning: a type of reasoning by which generalizations are drawn from patterns in observed data

inscribe: to draw inside

inscribed angle: an angle that has its vertex on a circle and is subtended by an arc of the circle

isosceles triangle: a triangle with two congruent sides

major arc: the larger arc of a circle defined by two points on the circle

minor arc: the smaller arc of a circle defined by two points on the circle

negation: the opposite of a statement, formed using the word *not*

obtuse triangle: a triangle with one obtuse angle

point of tangency: the point where a tangent touches a curve

polygon: a closed figure with three or more sides

postulate: an axiom

proof: a convincing argument

proof by contradiction: indirect proof

reflexive property: a property of equality that states, "For any a, $a = a$."

sector: a region bounded by two radii and an arc of a circle

sector angle: the central angle formed by a sector of a circle

statement: a sentence that is either true or false

subtend: to lie opposite to, as an arc subtends the opposite inscribed angle

tangent: a line that touches a curve at a single point

Tangent Chord Theorem: the theorem stating that the angle between a tangent and a chord of a circle equals the inscribed angle on the opposite side of the chord

theorem: a generalization that must be proven before being accepted as true

Venn diagram: a diagram of overlapping circles drawn within a rectangle used to model relationships among statements

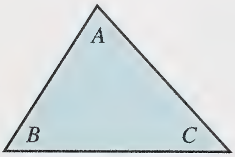
Suggested Answers

Section 1: Activity 1

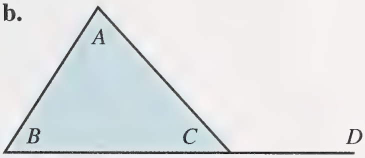
1. **a. Textbook questions a. to e. of "Explore: Perform an Experiment," p. 338**

Answers may vary. A sample experiment is given.

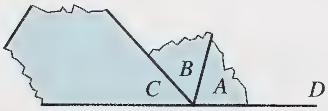
a.



b.

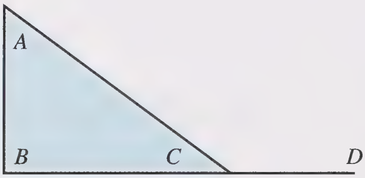


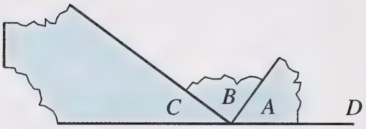
c. and d.



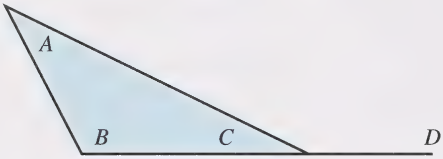
Section 1: Activity 1 (continued)

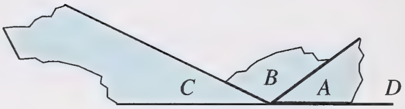
e. Perform the same experiment when $\angle B = 90°$.





Perform the same experiment when $\angle B > 90°$.





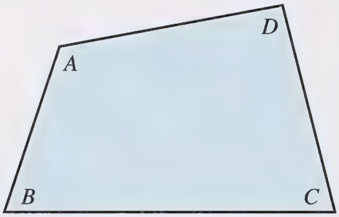
b. Textbook questions 1 to 7 of "Inquire," p. 338

1. The sum of the interior angles of a triangles is 180°.

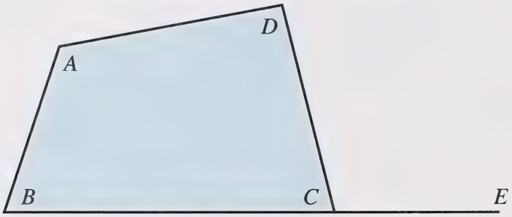
2. An exterior angle of a triangle equals the sum of the two interior and opposite angles.

3. Your classmates' conjectures should be similar to yours.

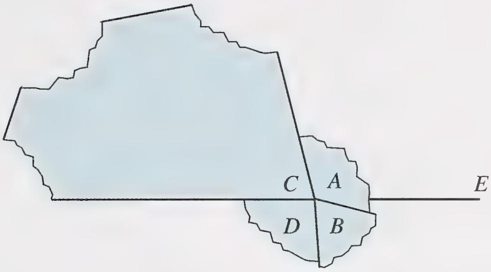
4. and 5. Draw quadrilateral *ABCD* on a piece of paper. Label the vertices inside the quadrilateral.



Extend *BC* to *E* to make exterior $\angle DCE$. Cut out the quadrilateral and the exterior angle.



Tear off $\angle A$, $\angle B$, and $\angle D$; and place them around $\angle C$ (as shown).



6. The sum of the interior angles of a quadrilateral is 360°.

7. **a.**
$$a + 40° + 70° = 180°$$
$$a + 110° = 180°$$
$$a = 70°$$

b.
$$b + 45° + 110° = 180°$$
$$b + 155° = 180°$$
$$b = 25°$$

c.
$$x + 2(90°) + 120° = 360°$$
$$x + 300° = 360°$$
$$x = 60°$$

d.
$$y + 80° + 70° + 115° = 360°$$
$$y + 265° = 360°$$
$$y = 95°$$

2. a. Textbook questions 1, 2, 7, 8, 9, and 10 of "Practice," pp. 340 and 341

1. The square of a number composed of n 1's consists of the digits 1 to n and $n-1$ to 1 in order.

$$11111^2 = 123\,454\,321$$
$$111111^2 = 12\,345\,654\,321$$

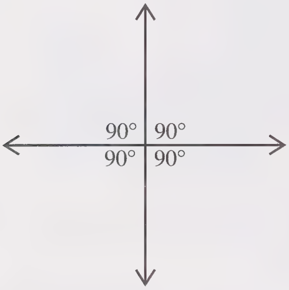
2. The sum of the first n odd numbers is equal to n^2.

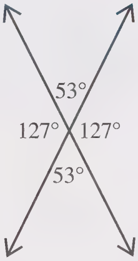
$$1+3+5+7+9 = 25$$
$$= 5^2$$ ← There are 5 odd numbers.

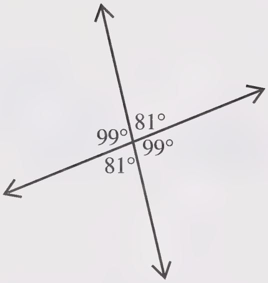
$$1+3+5+7+9+11 = 36$$
$$= 6^2$$ ← There are 6 odd numbers.

Section 1: Activity 1 (continued)

7. Diagrams may vary.

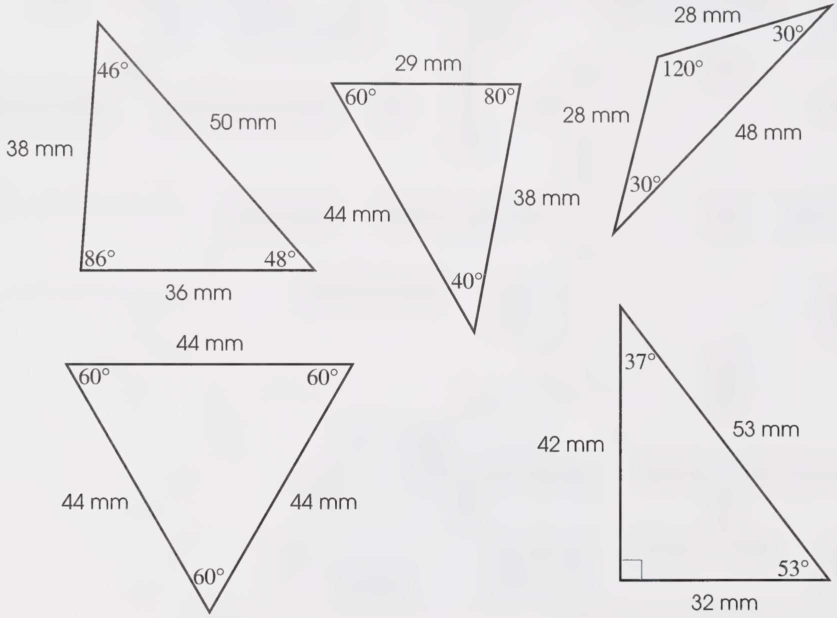






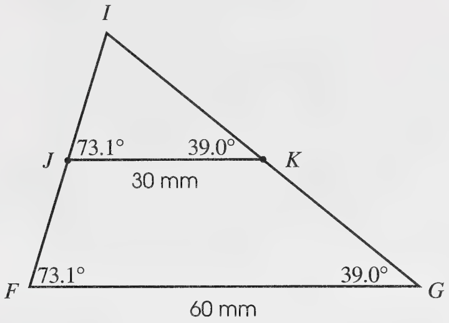
When two lines intersect at a point, the opposite angles are equal in measure.

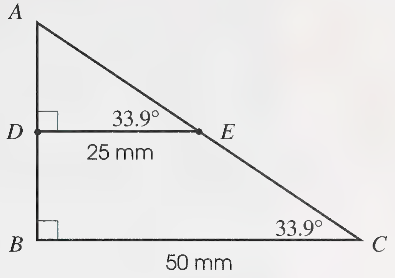
8.

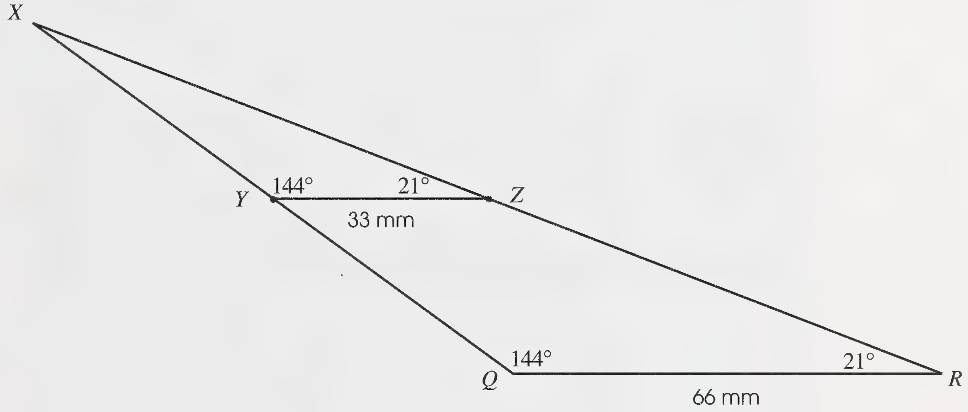


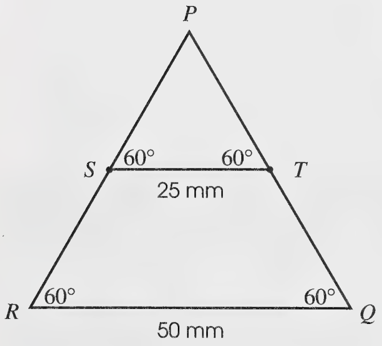
a. The location of the longest side is opposite the largest angle.
b. The location of the shortest side is opposite the smallest angle.

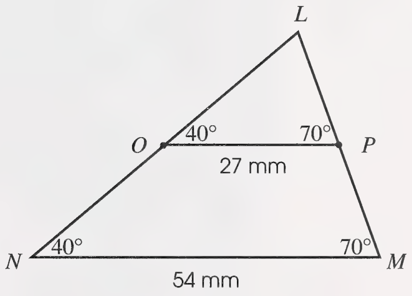
9. Your triangles may differ in shape.











Based on the diagrams, two conjectures are as follows:

i. The line segment joining the midpoints of two sides of a triangle is one-half the length of the third side of the triangle.

ii. The line segment joining the midpoints of two sides of a triangle is parallel to the third side of the triangle.

Section 1: Activity 1 (continued)

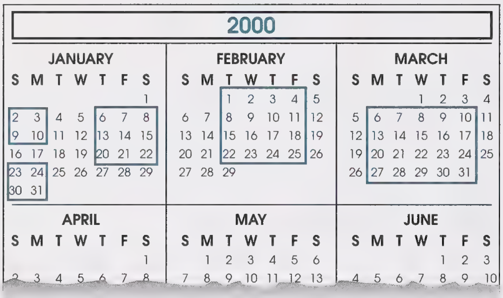
10. **a.** The sum of the numbers in the *n*th row is $(n-1)^3+n^3$.

b. Row 6: $26+27+28+29+30+31+32+33+34+35+36=341$

$$=5^3+6^3$$

b. Textbook questions 11, 12, 16, 21, and 22.a. of "Applications and Problem Solving," pp. 341 and 342

11. Answers may vary. A sample answer is given.



2000

JANUARY

S	M	T	W	T	F	S
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31					

FEBRUARY

S	M	T	W	T	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29				

MARCH

S	M	T	W	T	F	S
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

APRIL

S	M	T	W	T	F	S
						1
2	3	4	5	6	7	8

MAY

S	M	T	W	T	F	S
	1	2	3	4	5	6
7	8	9	10	11	12	13

JUNE

S	M	T	W	T	F	S
				1	2	3
4	5	6	7	8	9	10

For January 2000, the numbers at opposite corners in $\begin{array}{|cc|}\hline 2 & 3\\ 9 & 10\\ \hline\end{array}$ add up to 12.

$2+10=12$ and $3+9=12$

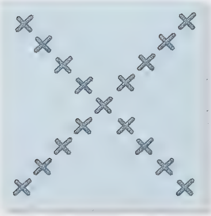
For any rectangular array of numbers that is at least 2×2, the sums of the two pairs of numbers at opposite corners are equal.

In January 2000, $\begin{array}{|ccc|}\hline 6 & 7 & 8\\ 13 & 14 & 15\\ 20 & 21 & 22\\ \hline\end{array}$ yields sums of $6+22=28$ and $20+8=28$. Similarly, $\begin{array}{|cc|}\hline 23 & 24\\ 30 & 31\\ \hline\end{array}$ yields $23+31=54$ and $30+24=54$.

In February 2000, the rectangular array yields sums of $1+25=26$ and $4+22=26$.

In March 2000, the rectangular array yields sums of $6+31=37$ and $10+27=37$.

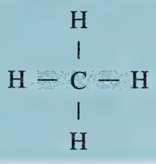
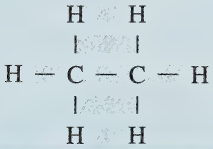
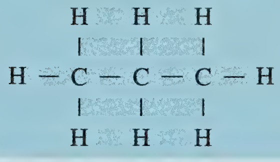
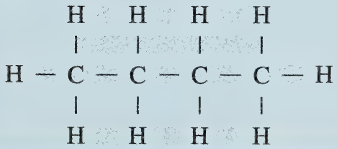
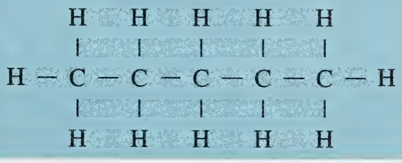
12.



Number of Stitches in a Diagonal	Total Number of Cross-Stitches
3	5
5	9
7	13
9	17
⋮	⋮
n	$2n-1$

If the number of cross-stitches in the diagonal is *n*, then the number of cross-stitches in the figure is $2n-1$.

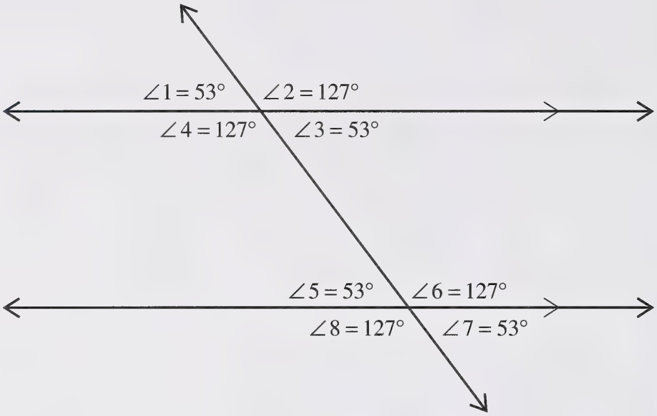
16. a.

Hydrocarbon	Carbon Atoms	Hydrogen Atoms
Methane (CH_4)  $\begin{array}{ccccc} & & H & & \\ & & \mid & & \\ H & - & C & - & H \\ & & \mid & & \\ & & H & & \end{array}$	1	4
Ethane (C_2H_6)  $\begin{array}{ccccccc} & & H & & H & & \\ & & \mid & & \mid & & \\ H & - & C & - & C & - & H \\ & & \mid & & \mid & & \\ & & H & & H & & \end{array}$	2	6
Propane (C_3H_8)  $\begin{array}{ccccccccc} & & H & & H & & H & & \\ & & \mid & & \mid & & \mid & & \\ H & - & C & - & C & - & C & - & H \\ & & \mid & & \mid & & \mid & & \\ & & H & & H & & H & & \end{array}$	3	8
Butane (C_4H_{10})  $\begin{array}{ccccccccccc} & & H & & H & & H & & H & & \\ & & \mid & & \mid & & \mid & & \mid & & \\ H & - & C & - & C & - & C & - & C & - & H \\ & & \mid & & \mid & & \mid & & \mid & & \\ & & H & & H & & H & & H & & \end{array}$	4	10
Pentane (C_5H_{12})  $\begin{array}{ccccccccccccc} & & H & & H & & H & & H & & H & & \\ & & \mid & & \mid & & \mid & & \mid & & \mid & & \\ H & - & C & - & C & - & C & - & C & - & C & - & H \\ & & \mid & & \mid & & \mid & & \mid & & \mid & & \\ & & H & & H & & H & & H & & H & & \end{array}$	5	12

b. In a straight-chain alkane, the number of hydrogen atoms is equal to two more than two times the number of carbon atoms.

Section 1: Activity 1 (continued)

21. Answers may vary. A sample answer is given.



a. The measures of corresponding angles are equal.

$$\angle 3 = \angle 7 = 53°$$

Other pairs of angles that are corresponding angles are as follows:

$$\angle 1 = \angle 5 = 53°$$
$$\angle 2 = \angle 6 = 127°$$
$$\angle 4 = \angle 8 = 127°$$

b. The co-interior angles are supplementary (add up to 180°).

$$\begin{aligned}\angle 3 + \angle 6 &= 53° + 127° \\ &= 180°\end{aligned}$$

Another pair of angles that are co-interior angles are $\angle 4$ and $\angle 5$.

$$\begin{aligned}\angle 4 + \angle 5 &= 127° + 53° \\ &= 180°\end{aligned}$$

22. a. $x \Delta y$ means multiply the first number by one more than the second number.

$$x \Delta y = x(y+1)$$

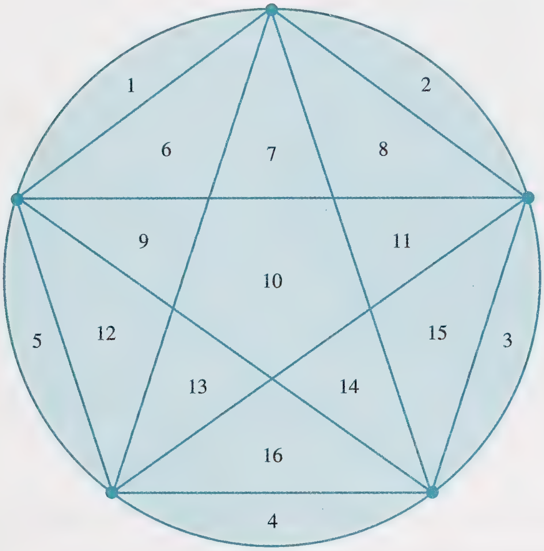
3. a. Textbook questions a. to c. of "Explore: Look for a Pattern," p. 343

a.

Points	Regions
2	2
3	4
4	8
5	16

There are 16 regions formed when 5 points are used.

b.



c. The number of regions is equal to 2^{n-1}, where *n* is the number of points.

b. Textbook questions 1 to 5 of "Inquire," p. 344

1. To test whether your formula is valid, use it to determine the number of regions for a various number of points. Check each test by drawing a diagram. Also, look for a counterexample to prove the formula is invalid.

Section 1: Activity 1 (continued)

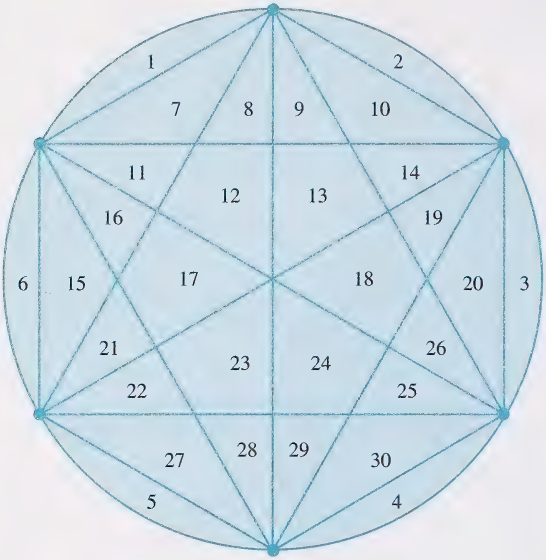
2. $2^{n-1} = 2^{6-1}$

$= 2^5$

$= 32$

There are 32 regions when 6 points are used.

3.



According to the diagram, only 30 regions are formed when 6 points are used.

4. **a.** Because a counterexample occurs when 6 points are used, the conjecture that the number of regions formed when *n* points are placed on the circle is 2^{n-1} is incorrect or invalid.

b. There may not be a simple relationship that you can use to predict the number of regions based on the number of points on the circle.

5. **a.** You could live in Mexico.
b. The quadrilateral could be a rectangle.
c. The vehicle could be a helicopter.
d. The number 9 is less than 10 and is not prime.

4. a. Textbook questions 1, 3, 4, 8, 9, and 10 of “Practice,” p 345

1. Answers may vary. A sample answer is given.

Two examples that support the conjecture are as follows:

- $32 \div 8 = 4$
- $48 \div 8 = 6$

A counterexample is $12 \div 8 = 1.5$, where 8 does not divide into 12 evenly.

3. Answers may vary. A sample answer is given.

Two examples that support the conjecture are as follows:

- $$\begin{aligned} 4^2 - 3^2 &= 16 - 9 \\ &= 7 \quad \longleftarrow \text{This is a prime number.} \end{aligned}$$
- $$\begin{aligned} 3^2 - 2^2 &= 9 - 4 \\ &= 5 \quad \longleftarrow \text{This is a prime number.} \end{aligned}$$

A counterexample is as follows:

$$\begin{aligned} 5^2 - 4^2 &= 25 - 16 \\ &= 9 \quad \longleftarrow \text{This is not a prime number.} \end{aligned}$$

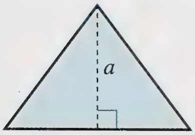
4. Two examples that support the conjecture are as follows:

- $\sqrt{25} = 5$ and $5 < 25$
- $\sqrt{36} = 6$ and $6 < 36$

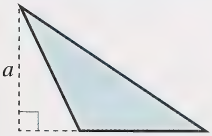
A counterexample is as follows:

$$\sqrt{\frac{1}{4}} = \frac{1}{2} \text{ and } \frac{1}{2} \not< \frac{1}{4}$$

8. Conjecture

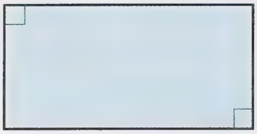


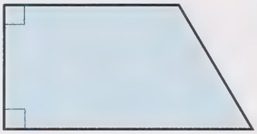
Counterexample



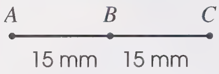
Section 1: Activity 1 (continued)

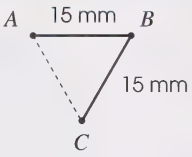
9. **Conjecture** **Counterexample**





10. **Conjecture** **Counterexample**



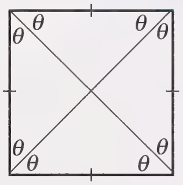


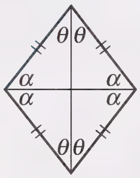
b. **Textbook questions 13, 14, 16, and 20 of "Applications and Problem Solving," p. 346**

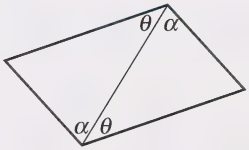
13. The provinces of Prince Edward Island and Newfoundland/Labrador do not share a border with the United States.

14. **a.** Neither the penguin nor the kiwi can fly.
b. Manx cats have no external tail.

16. **a.** **Conjectures** **Counterexample**







b. Danielle's conjecture is true only for those parallelograms with sides of equal length.

20. **a.** The following are two examples that support the conjecture that the sum and difference of two prime numbers are never both perfect squares.

- $7+2=9$ ← a perfect square
  $7-2=5$ ← not a perfect square
- $7+3=10$ ← not a perfect square
  $7-3=4$ ← a perfect square

One counterexample is as follows:

$2+2=4$ ← a perfect square
$2-2=0$ ← a perfect square, because $0^2=0$

b. There is only one counterexample.

Let p and q be prime numbers for which both the sum and difference are perfect squares.

$$\begin{aligned} \therefore\ p+q&=m^2 && \text{①} \\ p-q&=n^2 && \text{②} \\ \hline 2q&=m^2-n^2 && \text{①} - \text{②} \\ &=(m-n)(n+n) \end{aligned}$$

Now, if m and n are both even, then $m-n$ is even and $m+n$ is even. Because $(m-n)(m+n)$ is the product of two even numbers, q must be even. If q is even, then $q=2$, which is the only even prime.

Similarly, if m and n are both odd, then $m-n$ is even and $m+n$ is even. Again, $q=2$!

If one of m and n is even and the other is odd, then $m-n$ is odd and $m+n$ is odd. As a result, $(m-n)(m+n)$ is odd.

However, $(m-n)(m+n)=2q$, an even number!

This means there is only one possibility: $q=2$.

Recall that $m^2-n^2=2q$.

$$\begin{aligned} \therefore\ m^2-n^2&=2(2) \\ &=4 \end{aligned}$$

Consider the list of perfect squares.

$$\begin{array}{cccccccccccc} 0^2 & , & 1^2 & , & 2^2 & , & 3^2 & , & 4^2 & , & 5^2 & , \ldots \\ 0 & , & 1 & , & 4 & , & 9 & , & 16 & , & 25 & , \ldots \end{array}$$

There are only two perfect squares that differ by 4. They are 0^2 and 2^2. These are the values in the counterexample $2+2=2^2$ and $2-2=0^2$.

Section 1: Activity 1 (continued)

5. Textbook questions 1 and 2 of "PATTERN POWER," p. 346

1. For each column, the sum of the numbers in the first and second rows less the sum of the numbers in the third and fourth rows is the number in the fifth row.

2. $$\begin{aligned}(18+12)-(x+6)&=11\\30-x-6&=11\\24-x&=11\\13&=x\\x&=13\end{aligned}$$

   The missing number is 13.

Section 1: Activity 2

1. a. Textbook questions a. to d. of "Explore: Use Inductive and Deductive Reasoning," p. 347

a. Since $\angle GOH$ is a straight angle, $a+b=\mathbf{180°}$.
Since $\angle EOF$ is a straight angle, $c+b=\mathbf{180°}$.

b. $a+b=c+b$

c. $(a+b)-b=(c+b)-b$

d. $\therefore\ a=c$

b. Textbook questions 1 to 6 of "Inquire," p. 347

1. When two lines intersect, the opposite angles are equal.

2. Yes, you can be sure your conjecture is valid for all pairs of intersecting lines by using deductive reasoning. You must demonstrate that this conclusion follows logically from true statements.

3. Deductive reasoning is logical reasoning. If the hypothesis is true and there is no fault in the argument, then the conclusion must be true. Deductive reasoning does not rely on specific examples; it demonstrates the result in general.

4. Since $\angle EOF$ is a straight angle, $b+c=180^\circ$. Since $\angle GOH$ is a straight angle, $d+c=180^\circ$.

$$\begin{aligned} \therefore\ b+c&=d+c \\ (b+c)-c&=(d+c)-c \\ b&=d \end{aligned}$$

5. **a.** $a=65^\circ$ and $b=c=115^\circ$ **b.** $x=120^\circ$ and $y=z=60^\circ$ **c.** $p=20^\circ$ and $q=r=160^\circ$

6. Alice believed eating these cakes will change her size and that eating these cakes can't make her larger. Therefore, her conclusion, "Eating cakes must make me smaller," followed the statements she believed to be true.

2. a. Textbook questions 2, 4, 6, and 8 of "Practice," p. 349

2. All dogs have hearts.

4. The sum of 11 and 12 is an odd number.

6. The diagonals of rhombus *KLMN* intersect at right angles.

8. **a.**

	Trial 1	**Trial 2**	**Trial 3**	**Trial 4**
Choose a number:	4	7	5	9
Triple the number:	12	21	15	27
Add 2:	14	23	17	29
Add the original number:	18	30	22	38
Add 2:	20	32	24	40
Divide by 4:	5	8	6	10
Subtract 1:	4	7	5	9

The final number is the same as the original number.

b. To prove this is always true, let n be the original number.

Choose a number: n
Triple the number: $3n$
Add 2: $3n+2$
Add the original number: $3n+2+n=4n+2$
Add 2: $4n+2+2=4n+4$
Divide by 4: $(4n+4)\div 4=4(n+1)\div 4=n+1$
Subtract 1: $n+1-1=n$

Therefore, the final number is equal to the original number.

Section 1: Activity 2 (continued)

b. Textbook questions 9 to 13 and 18 of "Applications and Problem Solving," p. 349

9. a. Every odd number is of the form $2n+1$, where n is an integer.

Let $a=2n+1$, where n is an integer.
Let $b=2m+1$, where m is an integer.

$$\begin{aligned} \text{Now, } a+b &= (2n+1)+(2m+1) \\ &= 2n+2m+2 \\ &= 2(n+m+1) \end{aligned}$$

Therefore, $a+b$ is an even number because it is a multiple of 2.

b. Let $a=2n+1$, where n is an integer.
Let $b=2m+1$, where m is an integer.

$$\begin{aligned} \therefore\ ab &= (2n+1)(2m+1) \\ &= 4mn+2n+2m+1 \\ &= 2(2mn+n+m)+1 \end{aligned}$$

Let $p=2mn+n+m$.

$$\therefore\ ab=2p+1$$

Because ab is of the form $2p+1$, ab is an odd number.

10. Let n represent the smallest integer.

$$\begin{aligned} \therefore\ \text{Sum} &= n+(n+1)+(n+2) \\ &= 3n+3 \\ &= 3(n+1) \end{aligned}$$

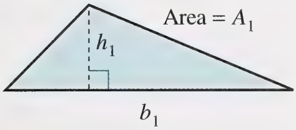
Because the sum $3(n+1)$ is a multiple of 3, the sum of three consecutive integers is a multiple of three.

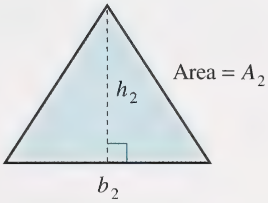
11. Let n and $n+1$ be the consecutive whole numbers.

$$\begin{aligned}\therefore\ (n+1)^2-n^2 &= \left(n^2+2n+1\right)-n^2\\ &=2n+1\\ &=n+(n+1)\end{aligned}$$

Therefore, the difference between the squares of consecutive whole numbers is equal to the sum of the two numbers.

12.





If $A_1=A_2$, then

$$\frac{1}{2}b_1h_1=\frac{1}{2}b_2h_2$$

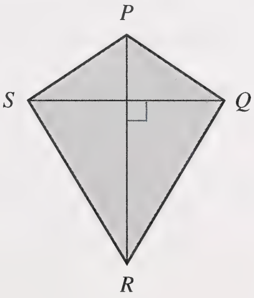
$$b_1h_1=b_2h_2$$

$$\frac{h_1}{h_2}=\frac{b_2}{b_1}$$

or $\dfrac{b_1}{b_2}=\dfrac{h_2}{h_1}$

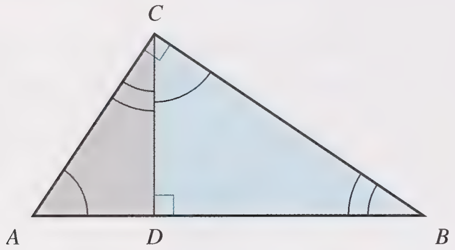
The ratio of height of the first triangle to the height of the second triangle equals the ratio of the base of the second triangle to the base of the first triangle.

13. Quadrilateral *PQRS* is a kite. For a rhombus or a square, the diagonals bisect each other; but for a kite, the diagonals do not bisect each other.



Section 1: Activity 2 (continued)

18. To prove that each of the smaller triangles is similar to the original triangle, show that their corresponding angles are equal.



For ΔABC and ΔACD,

$\angle BAC = \angle CAD$ ← Reflexive property

$\angle ACB = \angle ADC$ ← These are both right angles.

$\therefore\ \angle ABC = \angle ACD$ ← Triangle Angle Sum Theorem

For ΔABC and ΔBCD,

$\angle ABC = \angle CBD$ ← Reflexive property

$\angle ACB = \angle BDC$ ← These are both right angles.

$\therefore\ \angle BAC = \angle BCD$ ← Triangle Angle Sum Theorem

The symbol ~ means *is similar to.*

Therefore, $\Delta ABC \sim \Delta ACD$ and $\Delta ABC \sim \Delta BCD$.

3. a. Textbook questions a. to d. of "Explore: Compound Statements With *And* and *Or*," p. 350

Answers may vary. Sample answers are given.

a. Was Sir John A. Macdonald the first prime minister or a Father of Confederation?
b. Was that a mule deer or a white-tailed deer?
c. Either $a = b$ or $a \neq b$.
d. The inequality $4 < x < 6$ means $x < 6$ and $x > 4$.

b. Textbook questions 1 and 2 of "Inquire," p. 351

1. **a.** This is a statement that is false.
   b. This is a statement that is true.
   c. This is not a statement because it cannot be judged as true or false. This is an opinion.
   d. This is a statement that is false.
   e. This is a statement. It can be judged true or false if the values of x and y are known.
   f. This is a statement that is true.
   g. This is not a statement; it is an opinion.
   h. This is a statement that is true.

2. **a.** The use of *or* is exclusive because the same package cannot be delivered on both days.

   b. The use of *or* is inclusive because movies can be animated and musical.

   c. The use of *or* is exclusive because a car cannot be both entirely red and entirely white.

   d. This depends on the value of x. For example, if $x = 45$, the use of *or* is inclusive; and if $x = 25$, the use of *or* is exclusive.

   e. This depends on the value of y. For example, if $y = 5$, the use of *or* is inclusive; and if $y = 8$, the use of *or* is exclusive.

4. a. Textbook questions 1, 2, 6, 8, 12, 16, 18, 24, 31, 34, 36, 40, and 46 of "Practice, pp. 354 and 355

1. This is a statement that is true.

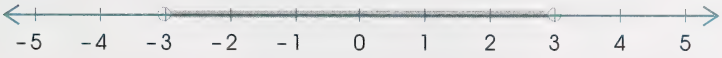
2. This is not a statement because it is not intended to be true or false.

6. If x is a factor of 18 and a factor of 30, then x is 1, 2, 3, or 6.

8. If x is an odd number and a factor of 12, then x is 1 or 3.

12. If y is a factor of 12 or if y is less than 10, then y is 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, or 12.

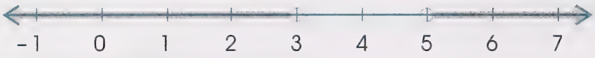
16.



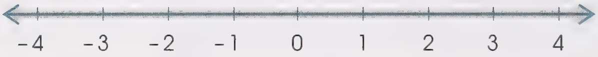
18. There is no solution because a number cannot be larger than 3 and less than or equal to -3 at the same time.

Section 1: Activity 2 (continued)

24.



31.

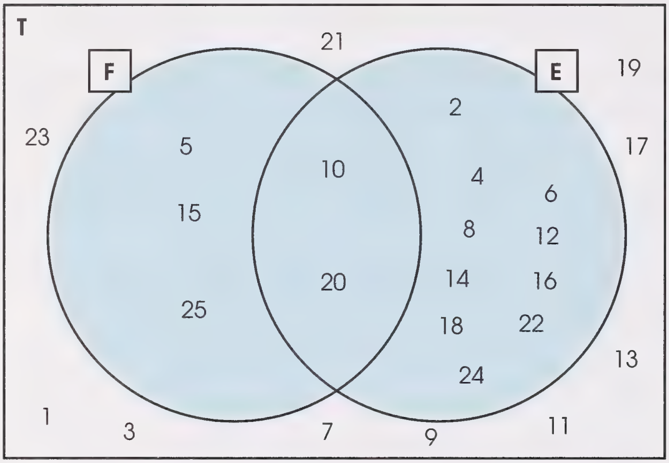


34. $x > -1$ and $x < 3$

36. $x < -2$ or $x \geq 2$

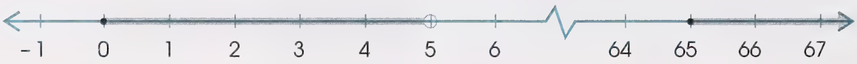
40. Paulo does not live in Edmonton.

46.



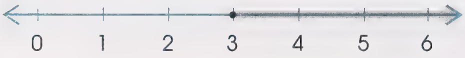
b. **Textbook questions 52.b., 53, 54.a., 54.c., 56, and 57 of "Applications and Problem Solving," pp. 355 and 356**

52. b.

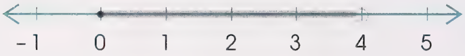


53. $n < 4$

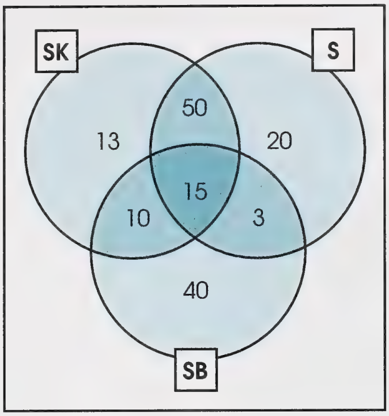
54. a.



c.



56. **a.**



$13+20+40+50+3+10+15=151$

The club has 151 members.

b. $50+15=65$

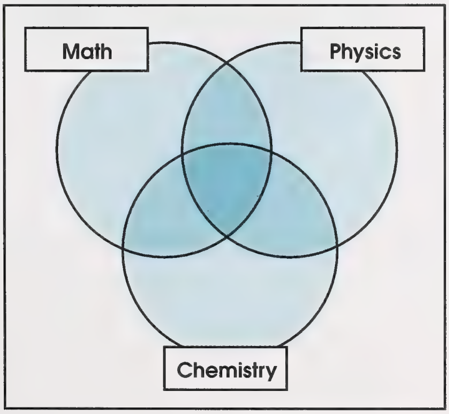
Therefore, 65 members ski and skate.

c. $20+3+40=63$

Therefore, 63 members do not ski.

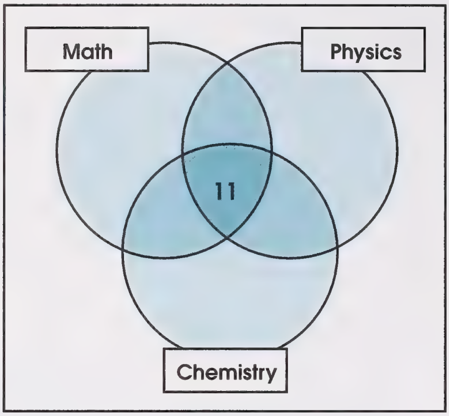
d. Only 13 members do not skate or snowboard.

57. **a.** **Step 1:** Draw a skeleton Venn Diagram.



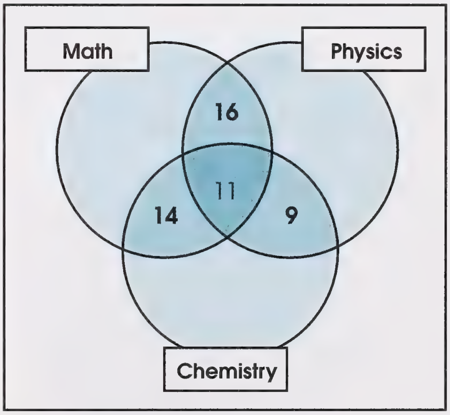
Section 1: Activity 2 (continued)

Step 2: Enter the number of students taking all three courses.



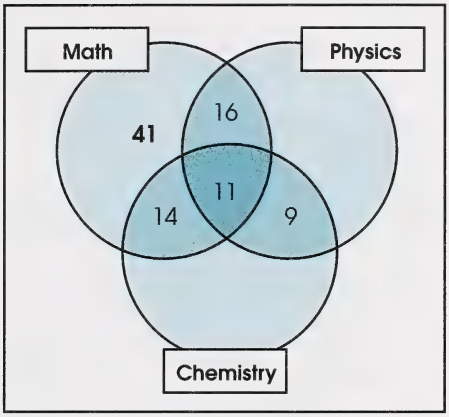
Step 3: Enter the amounts in the three other overlaps.

- There are $27-11=16$ students taking math and physics but not chemistry.
- There are $25-11=14$ students taking math and chemistry but not physics.
- There are $20-11=9$ students taking physics and chemistry but not math.



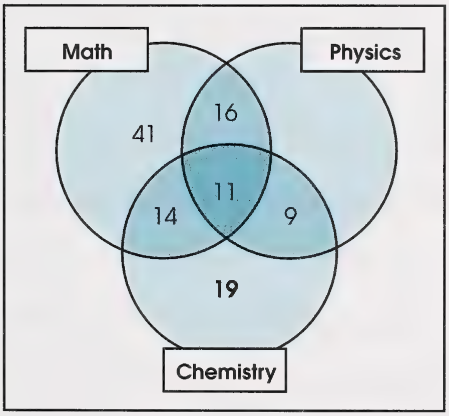
Step 4: Find the number of students taking only math.

$$\begin{aligned}82-(16+11+14)&=82-41\\&=41\end{aligned}$$



Step 5: Find the number of students taking only chemistry.

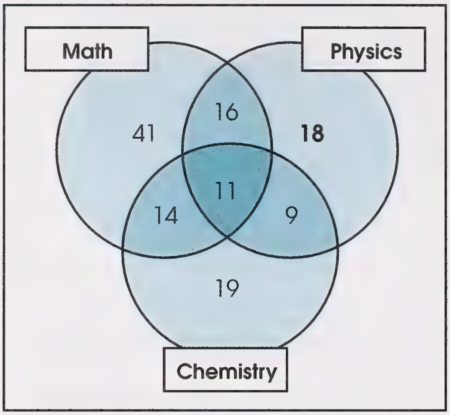
$$\begin{aligned}110-(41+16+11+14+9)&=110-91\\&=19\end{aligned}$$



Section 1: Activity 2 (continued)

Step 6: Find the number of students taking only physics.

$$\begin{aligned} 87-(16+11+14+19+9) &= 87-69 \\ &= 18 \end{aligned}$$



b. $41+14+16+11+18+9=109$

There are 109 students taking math or physics.

c. $\begin{aligned} 130-(41+16+18+14+11+9+19) &= 130-128 \\ &= 2 \end{aligned}$

There are 2 students taking none of these courses.

5. a. Textbook questions a. to g. of "Explore: Interpret the Statement," p. 359

- **a.** The hypothesis is "$x=2$."
- **b.** The conclusion is "$x^2=4$."
- **c.** The conditional statement is true.
- **d.** If $x^2=4$, then $x=2$.
- **e.** The hypothesis of the converse is "$x^2=4$."
- **f.** The conclusion of the converse is "$x=2$."
- **g.** The converse is not always true because $x=-2$ is also a true result.

b. Textbook questions 1 to 5 of "Inquire," p. 359

1. Answers may vary.

   The converse, "If $x^2 = 4$, then $x = 2$," is false.

   The counterexample is x can equal -2 because $(-2)^2 = 4$.

2. If $x < 2$, then $x^2 < 4$.

3. No, the conditional statement is false because $-8 < 2$ and $(-8)^2 = 64$, which is $\not< 4$.

4. If $x^2 < 4$, then $x < 2$.

5. Yes, the converse is true. If $x^2 < 4$, then $|x| < 2$, which also implies $x < 2$.

6. a. Textbook questions 2, 3, 4, 9, 12, 17, 18, 19, 23, and 26 of "Practice," pp. 361 and 362

2. If a Canadian is at least 18 years old, then that person may vote.

3. If a figure is a quadrilateral, then it is a polygon.

4. If a prime number is greater than 2, then it is an odd number.

9. A polygon is a pentagon if and only if it has exactly five sides.

12. A triangle is isosceles if and only if it has two sides of equal length.

17. If a rectangle is a square, then it has four equal sides.

    This converse is true because a square has four equal sides and four equal angles.

18. If a triangle is equilateral, then it has three equal sides.

    This converse is true because equilateral means *equal sided*.

19. If a quadrilateral is a trapezoid, then it has one pair of opposite sides that are parallel.

    This converse is true because it follows directly from the definition of a trapezoid.

Section 1: Activity 2 (continued)

23. The conditional statement is true.

converse: "If $2n+1$ is odd, then n is even."
contrapositive: "If $2n+1$ is not odd, then n is not even."

The converse is false. Two counterexamples are as follows:

$$\begin{aligned} 2n+1&=3 \\ 2n&=2 \\ n&=1 \end{aligned} \qquad\qquad \begin{aligned} 2n+1&=7 \\ 2n&=6 \\ n&=3 \end{aligned}$$

The contrapositive is true. Two examples follow:

$$\begin{aligned} 2n+1&=10 \\ 2n&=9 \\ n&=\frac{9}{2} \text{ or } 4.5 \end{aligned} \qquad\qquad \begin{aligned} 2n+1&=14 \\ 2n&=13 \\ n&=\frac{13}{2} \text{ or } 6.5 \end{aligned}$$

26. The conditional statement is true.

converse: "If a quadrilateral has equal diagonals, then it is a rectangle."
contrapositive: "If a quadrilateral does not have equal diagonals, then it is not a rectangle."

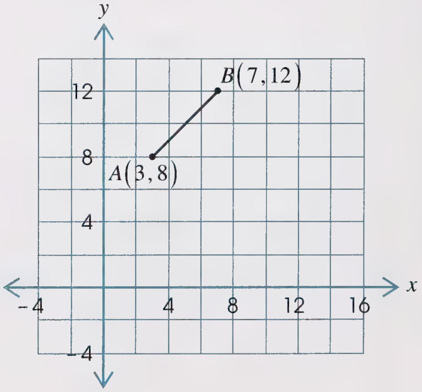
The converse is false, but the contrapositive is true.

b. Textbook questions 30 and 31 of "Applications and Problem Solving," p. 362

30. a. The conditional statement is true.

Let $(3,8)=(x_1,y_1)$ and $(7,12)=(x_2,y_2)$.

$$\begin{aligned} \therefore \text{ Midpoint of } AB &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \\ &= \left(\frac{3+7}{2}, \frac{8+12}{2}\right) \\ &= \left(\frac{10}{2}, \frac{20}{2}\right) \\ &= (5,10) \end{aligned}$$



b. converse: "If the midpoint of AB is $M(5,10)$, then the endpoints of AB are $A(3,8)$ and $B(7,12)$."

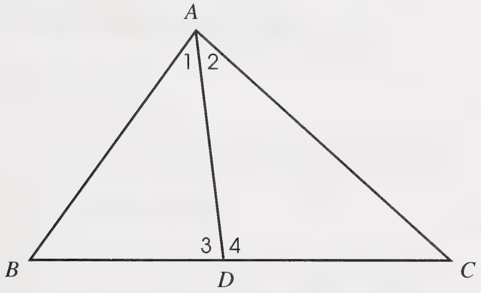
This conditional statement is false. One counterexample is that $M(5,10)$ is also the midpoint of the segment with endpoints $A(4,9)$ and $B(6,11)$.

31. Answers may vary. Sample answers are given.

a. The advertisers want you to think that you only have good taste if you eat at Tooleys.
b. The statement is not correct because eating at Tooleys does not ensure good taste.
c. Advertisers could emphasize that the food is tasty at Tooleys.

Section 1: Activity 3

1. a. Textbook question "Explore: Indirect Proof," p. 366



b. Textbook questions 1 to 8 of "Inquire," p. 366

1. $\angle 1 = \angle 2$ because AD bisects $\angle BAC$.

2. $\angle 3 = \angle 4$ because they are both right angles.

Remember: You are assuming $AD \perp BC$.

3. $\Delta ABD \cong \Delta ACD$ by ASA (angle-side-angle).

4. AB and AC are corresponding sides of the congruent triangles.

5. If $AB = AC$, then ΔABC is isosceles. This contradicts the assumption that ΔABC is scalene.

6. No, the bisector of an angle of a scalene triangle is not perpendicular to the base. If the triangle is isosceles, then the bisector of the angle between the equal sides is perpendicular to the base.

Section 1: Activity 3 (continued)

7. You want to prove that if $x^2 = 5x$, then $x \neq 2$. Assume $x = 2$.

   If $x = 2$, then it will satisfy $x^2 = 5x$.

LS	RS
x^2	$5x$
$=2^2$	$=5(2)$
$=4$	$=10$

   LS $\neq$ RS

   Because $\text{LS} \neq \text{RS}$, $x^2 \neq 5x$ when $x = 2$. This contradicts the statement $x^2 = 5x$.

   $\therefore\ x \neq 2$

8. Yes, the statement made by Sherlock Holmes illustrates indirect proof. Recall that a mathematical statement is either true or false. If you can show the opposite of what you want to prove is false or impossible, then the original statement must be true.

2. a. Textbook questions 1 to 5 of "Practice," p. 372

1. Assume that no two people in the group of 13 were born in the same month.

2. Assume there is a greatest whole number.

3. Assume that the sum of two odd integers is an odd integer.

4. Assume Suspect A is guilty of the crime.

5. **Prove**

   If the product of two natural numbers is greater than 100, then at least one of the numbers is greater than 10.

   Indirect Proof

   Assume that neither number is greater than 10.

   Let one of the natural numbers be 5.

   Therefore, the other natural number is $100 \div 5 = 20$.

   Because 20 is greater than 10, the assumption is false.

   Thus, at least one of the numbers is greater than 10.

b. Textbook questions 12 and 14 of "Applications and Problem Solving," p. 372

12. Prove

If x and y are positive integers such that their product is an odd number, then both x and y are odd numbers.

Indirect Proof

Assume that x and/or y is even.

If x is even, then $x = 2n$, where n is an integer.

$$\begin{aligned} \therefore\ xy &= 2n(y) \\ &= 2(ny) \end{aligned}$$

Therefore, xy is even because it is a multiple of 2. This contradicts the statement that xy is odd, making the assumption false.

Therefore, x and y are odd numbers.

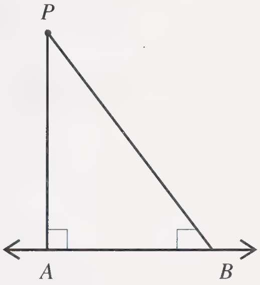
14. Prove

There is only one perpendicular from a point, P, to a line.

Indirect Proof

Assume there are two distinct perpendiculars, PA and PB, from point P to the line.

$$\begin{aligned} \therefore\ \angle PAB + \angle PBA + \angle APB &= 180^\circ \quad \longleftarrow \text{Triangle Angle Sum Theorem} \\ 90^\circ + 90^\circ + \angle APB &= 180^\circ \\ 180^\circ + \angle APB &= 180^\circ \\ \angle APB &= 0^\circ \end{aligned}$$



But, because PAB is a triangle, $\angle APB$ must be greater than 0°.

$$\therefore\ \angle PAB + \angle PBA + \angle APB > 180^\circ$$

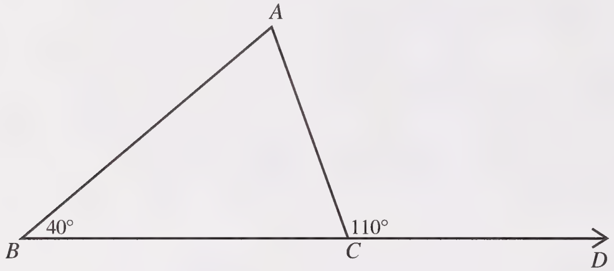
This contradicts the Triangle Angle Sum Theorem, making the assumption false.

Therefore, there is only one perpendicular from a point, P, to a line.

Section 1: Activity 3 (continued)

3. Textbook questions 16, 18, and 19 of "Applications and Problem Solving," p. 372

16. Given $\angle ACD = 110°$ and $\angle B = 40°$, prove that ΔABC is isosceles.



Direct Proof

$\angle BCD$ is a straight angle.

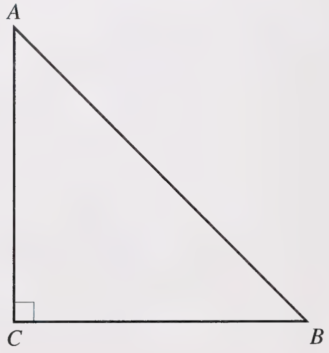
$$\begin{aligned} \therefore\ \angle ACB + \angle ACD &= 180° \\ \angle ACB + 110° &= 180° \\ \angle ACB &= 70° \end{aligned}$$

$$\begin{aligned} \therefore\ \angle A + \angle B + \angle ACB &= 180° \\ \angle A + 40° + 70° &= 180° \\ \angle A &= 70° \end{aligned}$$

← Triangle Angle Sum Theorem

Because $\angle A = \angle ACB$, ΔABC is isosceles.

18. Given ΔABC, where $\angle C = 90°$, prove that $\angle A + \angle B = 90°$.



Direct Proof

$$\begin{aligned} \angle A + \angle B + \angle C &= 180° \\ \angle A + \angle B + 90° &= 180° \\ \angle A + \angle B &= 90° \end{aligned}$$

Therefore, the sum of the acute angles in a right triangle is 90°.

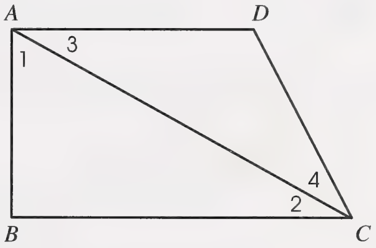
19. Given quadrilateral *ABCD*, prove $\angle A+\angle B+\angle C+\angle D=360^\circ$.

Divide *ABCD* into two triangles by drawing diagonal *AC*.

Therefore, $\angle 1+\angle 2+\angle B=180^\circ$ and $\angle 3+\angle 4+\angle D=180^\circ$

$$\therefore\ \angle 1+\angle 2+\angle B+\angle 3+\angle 4+\angle D=180^\circ+180^\circ$$
$$\angle 1+\angle 3+\angle B+\angle 2+\angle 4+\angle D=360^\circ$$
$$\angle A+\angle B+\angle C+\angle D=360^\circ$$

Therefore, the angle sum of any convex quadrilateral is 360°.



$\angle A=\angle 1+\angle 3$

$\angle C=\angle 2+\angle 4$

4. Textbook question "NUMBER POWER," p. 372

The sum of the first column is $3x+6$. The sum of the third column is $4x+2$.

$x+5$	$x-2$	$x+3$
x	$x+2$	$x+4$
$x+1$	$x+6$	$2x-5$

Because this is a magic square,

$$3x+6=4x+2$$
$$3x=4x-4$$
$$-x=-4$$
$$x=4$$

Substitute $x=4$ into each cell.

9	2	7
4	6	8
5	10	3

Therefore, the only number from 1 to 10 that is not used is 1.

Section 1: Activity 4

1. a. **Textbook questions a. to f. of "Explore: Complete the Proof," p. 385**

   - **a.** given
   - **b.** given
   - **c.** Opposite Angles Theorem
   - **d.** SAS
   - **e.** congruent triangles
   - **f.** Transversal Parallel Lines Theorem

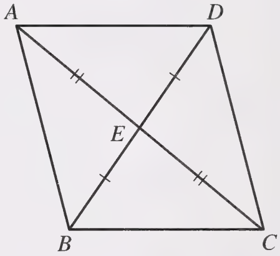
   b. **Textbook questions 1 to 3 of "Inquire," p. 386**

   1. $\angle DAE = \angle BCE$ because these two angles lie opposite congruent sides in the congruent triangles.

   2. $\angle ADE$ and $\angle CBE$, which are alternate angles, could have also been used to prove that AD and BC are parallel by the Transversal Parallel Lines Theorem.

   3. **Given:** $AE = CE$; $ED = EB$

   Prove: $AB = CD$ and $AB \parallel CD$



Proof:

Statements	Reasons
In ΔAEB and ΔCED,	
$AE = CE$	Given
$EB = ED$	Given
$\angle AEB = \angle CED$	Opposite Angles Theorem
$\Delta AEB \cong \Delta CED$	SAS
$AB = CD$	Congruent triangles
$\angle BAE = \angle DCE$	Congruent triangles
$AB \parallel CD$	Transversal Parallel Lines Theorem

2. a. **Textbook questions 1, 2, 3, 4, and 9 of "Practice," pp. 389 and 390**

1.

Statements	**Reasons**
In ΔABD and ΔACD,	
$BD = CD$	Definition of midpoint
$\angle BDA = \angle CDA$	**Definition of perpendicular**
$AD = AD$	**Reflexive property**
$\Delta ABD \cong \Delta ACD$	**SAS**
$\angle B = \angle C$	Congruent triangles

2.

Statements	**Reasons**
$BD = CD$	**Given**
$\angle DBC = \angle DCB$	Isosceles Triangle Theorem
$\angle ABD = \angle ACD$	**Given**
$\angle ABD + \angle DBC = \angle ACD + \angle DCB$	**Addition property**
$\angle ABC = \angle ACB$	Substitution property
$AB = AC$	**Isosceles Triangle Theorem**

3.

Statements	**Reasons**
In ΔPQT and ΔRST,	
$QT = ST$	**Given**
$\angle PTQ = \angle RTS$	Opposite Angles Theorem
$\angle P = \angle R$	**Transversal Parallel Lines Theorem**
$\Delta PQT \cong \Delta RST$	AAS
$TP = TR$	**Congruent triangles**

Section 1: Activity 4 (continued)

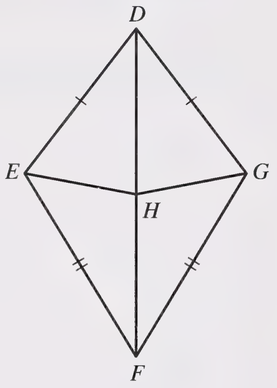
4.

Statements	**Reasons**
In ΔAOC and ΔBOC,	
$\angle OAC = \angle OBC = 90^\circ$	**Given**
$\boldsymbol{OC = OC}$	Reflexive property
$OA = OB$	**Radii of a circle**
$\Delta AOC \cong \Delta BOC$	**Hypotenuse-Side Congruence Theorem (HS)**
$AC = BC$	**Congruent triangles**

9. **Given:** $DE = DG$

$FE = FG$

Prove: $EH = GH$

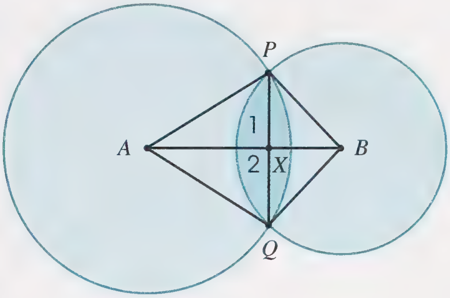


Proof:

Statements	**Reasons**
In ΔDEF and ΔDGF,	
$DE = DG$	Given
$FE = FG$	Given
$DF = DF$	Reflexive property
$\Delta DEF \cong \Delta DGF$	SSS
$\angle EDH = \angle GDH$	Congruent triangles
In ΔEDH and ΔGDH,	
$DE = DG$	Given
$\angle EDH = \angle GDH$	Proved
$DH = DH$	Reflexive property
$\Delta EDH \cong \Delta GDH$	SAS
$EH = GH$	Congruent triangles

b. Textbook questions 16, 22, and 26 of "Applications and Problem Solving," pp. 391 and 392

16. Given: Two circles, with centres A and B, intersect at P and Q.



Prove: $PQ \perp AB$

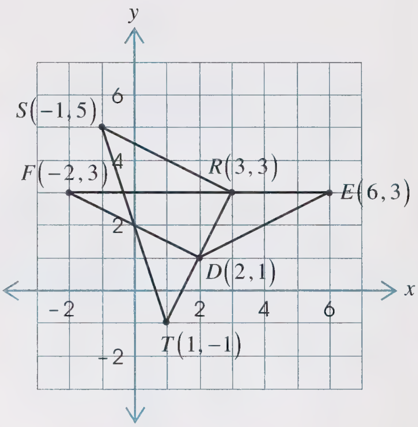
Proof:

Statements	Reasons
In ΔPAB and ΔQAB,	
$AP = AQ$	Radii of a circle
$BP = BQ$	Radii of a circle
$AB = AB$	Reflexive property
$\Delta PAB \cong \Delta QAB$	SSS
$\angle PAB = \angle QAB$	Congruent triangles
In ΔPAX and ΔQAX,	
$AP = AQ$	Radii of a circle
$\angle PAB = \angle QAB$	Proved
$AX = AX$	Reflexive property
$\Delta PAX \cong \Delta QAX$	SAS
$\angle 1 = \angle 2$	Congruent triangles
$\angle 1 + \angle 2 = 180°$	$\angle PXQ$ is a straight angle.
$\angle 1 = \angle 2 = 90°$	Equal supplementary angles
$PQ \perp AB$	$\angle 1$ is a right angle.

22. $\angle CEB$ must be a straight angle, where $CE = BE$; $\angle DEA$ must be a straight angle, where $DE = AE$. Therefore, $CD = AB$.

Section 1: Activity 4 (continued)

26. To determine if $\Delta RST \cong \Delta DEF$, compare the lengths of their sides.



For ΔRST,

$$\begin{aligned}RS &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\ &= \sqrt{(-1-3)^2+(5-3)^2}\\ &= \sqrt{(-4)^2+(2)^2}\\ &= \sqrt{16+4}\\ &= \sqrt{20}\\ &= 2\sqrt{5}\end{aligned}$$

$$\begin{aligned}RT &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\ &= \sqrt{(1-3)^2+(-1-3)^2}\\ &= \sqrt{(-2)^2+(-4)^2}\\ &= \sqrt{4+16}\\ &= \sqrt{20}\\ &= 2\sqrt{5}\end{aligned}$$

$$\begin{aligned}ST &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\ &= \sqrt{[1-(-1)]^2+(-1-5)^2}\\ &= \sqrt{(2)^2+(-6)^2}\\ &= \sqrt{4+36}\\ &= \sqrt{40}\\ &= 2\sqrt{10}\end{aligned}$$

For ΔDEF,

$$\begin{aligned}EF &= |x_2-x_1|\\ &= |-2-6|\\ &= |-8|\\ &= 8\end{aligned}$$

Because EF is not the same length as any of the sides in ΔRST, $\Delta RST \ncong \Delta DEF$.

3. **Textbook questions 1 to 4 of "Mental Math: Angle Measures," p. 383**

1. $b = 75°$
$a = 30°$

2. $a = 55°$
$c = 50°$
$b = 75°$

3. $p = 70°$
$q = 110°$
$r = 70°$
$s = 110°$

4. $a = 30°$
$b = 70°$
$c = 70°$

Section 1: Follow-up Activities

Extra Help

1. **Textbook questions 1 to 4 of Investigation 1, "Congruent Triangles," p. 364**

1. a. $AB = PQ$
$\angle B = \angle Q$
$BC = QR$

b. $\Delta ABC \cong \Delta PQR$ by SAS.

c. $AC = PR$
$\angle A = \angle P$
$\angle C = \angle R$

2. a. $\angle D = \angle J$
$DF = JL$
$\angle F = \angle L$

b. $\Delta DEF \cong \Delta JKL$ by ASA.

c. $\angle E = \angle K$
$DE = JK$
$EF = KL$

3. a. $XZ = RT$
$\angle X = \angle R$
$XY = RS$

b. $\Delta XYZ \cong \Delta RST$ by SAS.

c. $YZ = ST$
$\angle Y = \angle S$
$\angle Z = \angle T$

4. a. $PR = UV$
$PQ = UW$
$QR = VW$

b. $\Delta PRQ \cong UVW$ by SSS.

c. $\angle P = \angle U$
$\angle Q = \angle W$
$\angle R = \angle V$

Extra Help (continued)

2. Textbook questions 1, 3, and 5 of Investigation 2, "Triangles with Common Points and Sides," p. 365

1. a. $AB = CD$
$AC = AC$
$BC = AD$

b. $\Delta ABC \cong \Delta ADC$ by SSS.

c. $\angle BAC = \angle DCA$
$\angle B = \angle D$
$\angle BCA = \angle DAC$

3. a. $XZ = XZ$
$\angle WZX = \angle YZX$
$WZ = YZ$

b. $\Delta WXZ \cong \Delta XYZ$ by SAS.

c. $WX = XY$
$\angle W = \angle Y$
$\angle WXZ = \angle YXZ$

5. a. $\angle DAC = \angle BCA$
$AC = AC$
$\angle BAC = \angle ACD$

b. $\Delta ADC \cong \Delta ABC$ by ASA.

c. $\angle B = \angle D$
$AB = CD$
$BC = AD$

Enrichment

1. a. $1+1=2$ is true; and $1+2 \neq 2$ is true.

Therefore, "If $1+1=2$, then $1+2 \neq 2$," is true.

b. $1+1=3$ is false; and $1+2=4$ is false.

Therefore, "If $1+1=3$, then $1+2=4$," is true.

c. $1+1=2$ is true; and $1+2=2$ is false.

Therefore, "If $1+1=2$, then $1+2=2$," is false.

d. $1+1=3$ is false; and $1+2=3$ is true.

Therefore, "If $1+1=3$, then $1+2=3$," is true.

2.

p	q	$p \rightarrow q$	$\sim q$	$\sim p$	$\sim q \rightarrow \sim q$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Since the columns for $p \rightarrow q$ and $\sim q \rightarrow \sim p$ are identical, a conditional statement and its contrapositive are logically equivalent.

3.

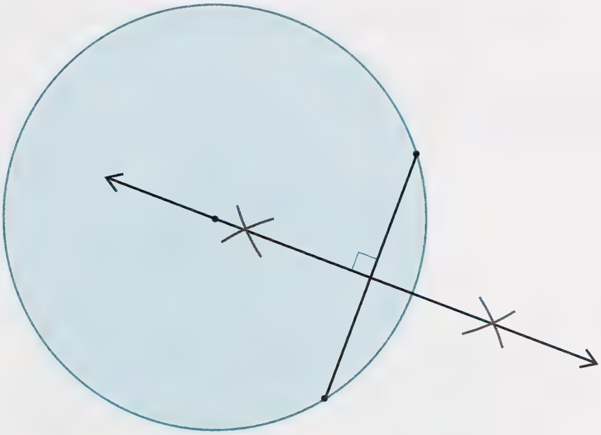
p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Because the columns for $p \rightarrow q$ and $q \rightarrow p$ are not the same, a conditional statement and its converse are not logically equivalent.

Section 2: Activity 1

1. a. **Textbook questions 1 to 7 of Investigation 1, "Perpendicular Bisector of a Chord," p. 395**

   1. Drawings may vary. A sample drawing is given.



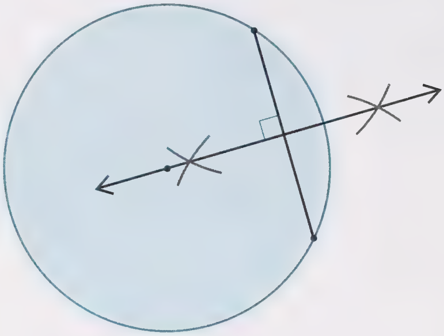
   2. Refer to the answer to question 1.

   3. Refer to the answer to question 1.

   4. The centre of the circle lies on the perpendicular bisector of the chord.

Section 2: Activity 1 (continued)

5. Drawings may vary. A sample drawing is given.

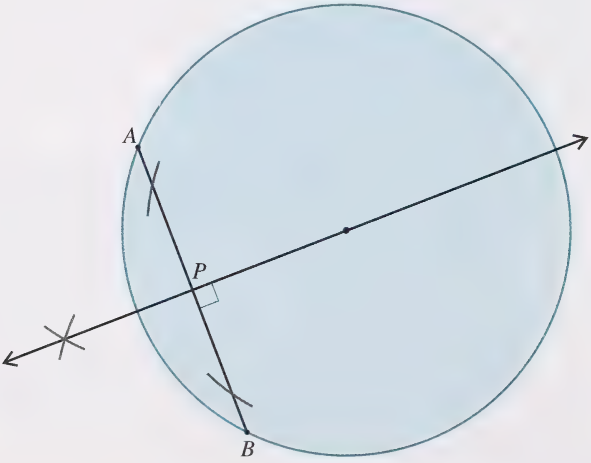


6. Your results should be the same as those of your classmates.

7. The centre of a circle lies on the perpendicular bisector of a chord.

b. Textbook questions 1 to 8 of Investigation 2, "Perpendicular From the Centre of a Circle to a Chord," p. 395

1. Drawings may vary. A sample drawing is given.



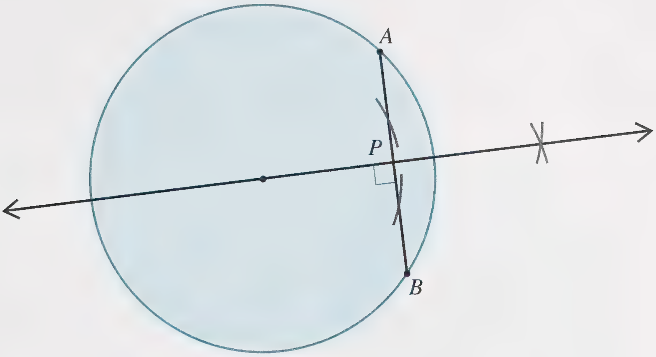
2. Refer to the answer to question 1.

3. Refer to the answer to question 1.

4. $AP = BP = 23$ mm

5. The distances are equal.

6. Drawings may vary. A sample drawing is given.



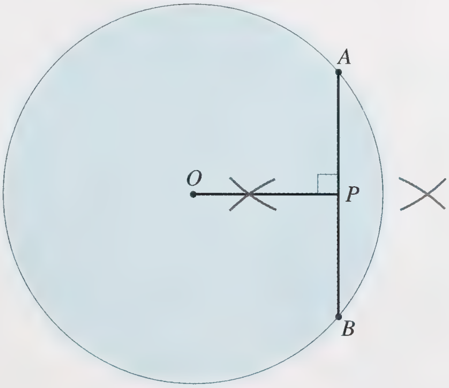
$AP = BP = 17$ mm

7. Your results should be the same as those of your classmates.

8. The perpendicular from the centre of a circle to a chord bisects the chord.

c. Textbook questions 1 to 9 of Investigation 3, "Line Segment From the Centre of a Circle to the Midpoint of a Chord," p. 395

1. Drawings may vary. A sample drawing is given.



2. Refer to the answer to question 1.

3. Refer to the answer to question 1.

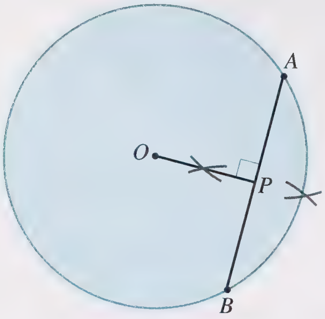
Section 2: Activity 1 (continued)

4. Refer to the answer to question 1.

5. $\angle OPA = \angle OPB = 90°$

6. The angles are right angles.

7. Drawings may vary. A sample drawing is given.



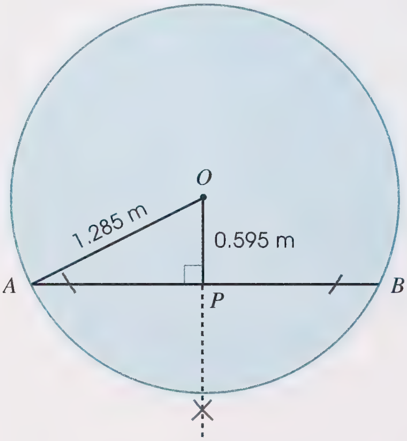
$\angle OPA = \angle OPB = 90°$

8. Your results should be the same as those of your classmates.

9. The line segment joining the centre of a circle and the midpoint of a chord is perpendicular to the chord.

2. a. Textbook question "Explore: Use a Diagram," p. 396

Answers may vary.



$$\begin{aligned} OP &= 1.88 - 1.285 \\ &= 0.595 \text{ m} \end{aligned}$$

b. Textbook questions 1 to 4 of "Inquire," p. 396

1. A right triangle, ΔOPA was drawn.
2. The Pythagorean Theorem can be used to relate the side lengths of this triangle.
3. $OA = 1.285$ m and $OP = 0.595$ m

$$\begin{aligned}\therefore AP &= \sqrt{(AO)^2 - (OP)^2}\\ &= \sqrt{1.285^2 - 0.595^2}\\ &\doteq 1.14 \text{ m}\end{aligned}$$

4. The width of the floor is $2 \times 1.14 \doteq 2.28$ m.

3. a. Textbook questions 1 to 6, 10, 13, and 14 of "Practice," p. 400

1. Yes, $AC = CB$ because the perpendicular drawn from the centre of a circle to a chord bisects the chord.
2. Yes, $\angle OCA = \angle OCB = 90°$ because the line segment joining the centre of a circle and the midpoint of a chord is perpendicular to the chord.
3. No, $AC \neq CB$ because the perpendicular to the chord does not pass through the centre of the circle.
4. No, $AB \neq FE$ because the chords are not drawn the same distance from the centre of the circle. In the diagram, $OB \neq OE$.
5. **a.** $$\begin{aligned}AO &= \sqrt{3^2 + 5^2} \quad \longleftarrow \text{Pythagorean Theorem}\\ &= \sqrt{9 + 25}\\ &= \sqrt{34}\\ &\doteq 5.8 \text{ cm}\end{aligned}$$

   b. $$\begin{aligned}AB &= 2(AC)\\ &= 2(5)\\ &= 10 \text{ cm}\end{aligned}$$

Section 2: Activity 1 (continued)

6.
$$\begin{aligned} OC &= 23-13 \\ &= 10 \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(OB)^2-(OC)^2} && \longleftarrow \text{Pythagorean Theorem} \\ &= \sqrt{23^2-10^2} \\ &= \sqrt{529-100} \\ &= \sqrt{429} \\ &\doteq 20.7 \text{ cm} \end{aligned}$$

$$\begin{aligned} AB &= 2(BC) \\ &\doteq 2(20.7) \\ &\doteq 41.4 \text{ cm} \end{aligned}$$

10. **a.**
$$\begin{aligned} (OC)^2+(BC)^2 &= (OB)^2 && \longleftarrow \text{Pythagorean Theorem} \\ (OC)^2+(OC)^2 &= (OB)^2 && \longleftarrow OC = BC \\ 2(OC)^2 &= (12)^2 \\ 2(OC)^2 &= 144 \\ (OC)^2 &= 72 \\ OC &\doteq 8.5 \text{ cm} \end{aligned}$$

b.
$$\begin{aligned} AB &= 2(BC) \\ &\doteq 2(8.5) && \longleftarrow BC = OC \\ &\doteq 17.0 \text{ cm} \end{aligned}$$

13.
$$\begin{aligned} OE &= \sqrt{(OC)^2-(CE)^2} && \longleftarrow \text{Pythagorean Theorem} \\ &= \sqrt{8^2-5^2} && \longleftarrow OC = AO \text{ and } CE = \tfrac{1}{2}CO \\ &= \sqrt{64-25} \\ &= \sqrt{39} \\ &\doteq 6.2 \text{ cm} \end{aligned}$$

14. a. $OB = OC + CD$
$= 2 + 3$
$= 5$

$$\begin{aligned} BC &= \sqrt{(OB)^2 - (OC)^2} \quad \longleftarrow \text{Pythagorean Theorem} \\ &= \sqrt{5^2 - 3^2} \\ &= \sqrt{25 - 9} \\ &= \sqrt{16} \\ &= 4 \text{ cm} \end{aligned}$$

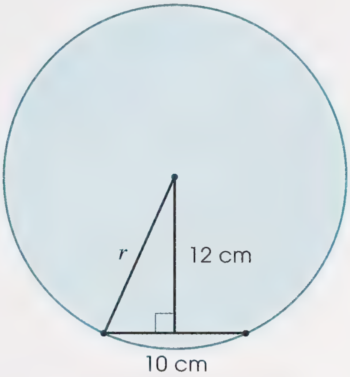
$$\begin{aligned} BD &= \sqrt{(BC)^2 + (CD)^2} \quad \longleftarrow \text{Pythagorean Theorem} \\ &= \sqrt{4^2 + 2^2} \\ &= \sqrt{16 + 4} \\ &= \sqrt{20} \\ &\doteq 4.5 \text{ cm} \end{aligned}$$

b. Textbook questions 17, 19, 21, 22, 26, and 30 of "Applications and Problem Solving," pp. 400 to 401

17. Using the Pythagorean Theorem,

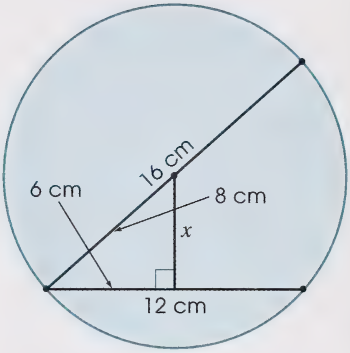
$$\begin{aligned} r &= \sqrt{5^2 + 12^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

The radius is 13 cm.



Section 2: Activity 1 (continued)

19. Let x be the distance from the centre of the circle to the chord.

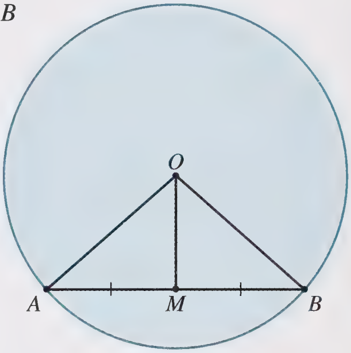


Using the Pythagorean Theorem,

$$\begin{aligned} x &= \sqrt{8^2 - 6^2} \\ &= \sqrt{64 - 36} \\ &= \sqrt{28} \\ &\doteq 5.3 \end{aligned}$$

The chord is about 5.3 cm from the centre.

21. Given: Circle with centre O; chord AB; M is the midpoint of AB

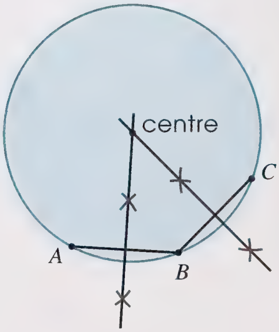


Prove: $OM \perp AB$

Proof:

Statements	**Reasons**
In ΔAOM and ΔBOM,	
$AO = BO$	Radii of a circle
$OM = OM$	Reflexive property
$AM = BM$	Definition of a midpoint
$\Delta AOM \cong \Delta BOM$	SSS
$\angle AMO = \angle BMO$	Congruent triangles
$\angle AMO + \angle BMO = 180°$	$\angle AMB$ is a straight angle.
$\angle AMO = \angle BMO = 90°$	$\angle AMO$ and $\angle BMO$ are equal and supplementary.
$OM \perp AB$	$\angle AMO$ is a right angle.

22. To locate the centre of the circle, perform the following steps:

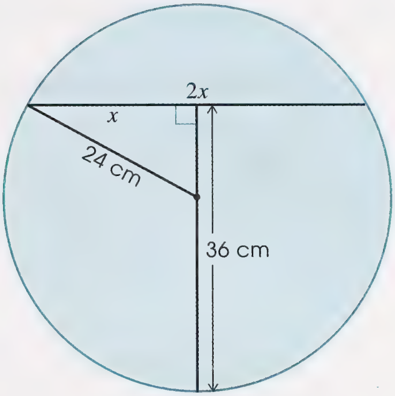


Step 1: Construct the perpendicular bisector of AB.

Step 2: Construct the perpendicular bisector of BC.

Step 3: Locate the point of intersection of the two perpendicular bisectors. This is the centre of the circle.

26. Let $2x$ be the width of the water surface.



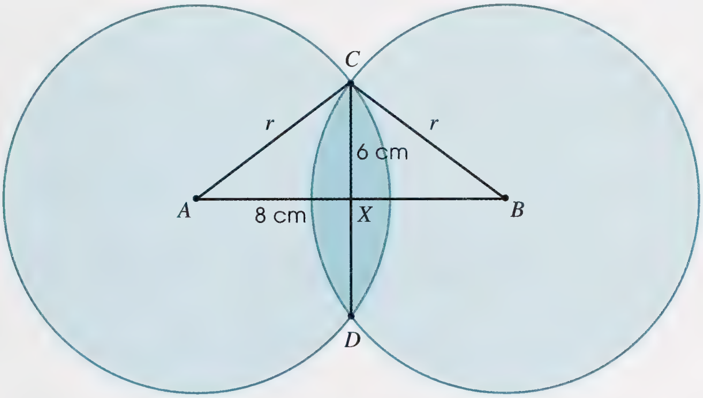
Using the Pythagorean Theorem,

$$\begin{aligned} x &= \sqrt{24^2 - 12^2} \\ &= \sqrt{576 - 144} \\ &= \sqrt{432} \\ &\doteq 20.8 \end{aligned}$$

12 = 36 – 24

The width of the water surface is about $2(20.8\text{ cm})$ or 41.6 cm. In this question, you assumed that the waterpipe is horizontal.

30.



If $AB = 16$ cm, then $AX = 8$ cm.
If $CD = 12$ cm, then $CX = 6$ cm.

$$\begin{aligned} \therefore\ r &= \sqrt{6^2 + 8^2} \quad \longleftarrow \text{Pythagorean Theorem} \\ &= \sqrt{36 + 64} \\ &= \sqrt{100} \\ &= 10 \end{aligned}$$

The radius of each circle is 10 cm.

Section 2: Activity 1 (continued)

4. a. Textbook questions 11 and 15 of "Practice," p. 400

11. a. $AB = \sqrt{6^2 - 3^2}$ ← Pythagorean Theorem

$$\begin{aligned} &= \sqrt{36-9} \\ &= \sqrt{27} \\ &\doteq 5.2 \text{ cm} \end{aligned}$$

b. Because chords *AC* and *FD* are equidistant from the centre of the circle,

$$\begin{aligned} FD &= AC \\ &= 2(AB) \\ &\doteq 2(5.2) \\ &\doteq 10.4 \text{ cm} \end{aligned}$$

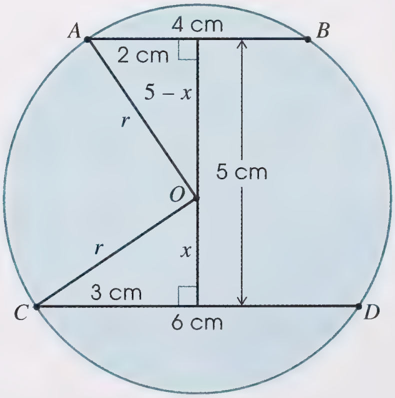
15. Because $\Delta OCD \cong \Delta OAB$ by SSS, $\angle OCD$ and $\angle OAB$ are corresponding angles of congruent triangles. Therefore, $\angle OCD = 29°$.

b. Textbook questions 24, 32, and 34 of "Applications and Problem Solving," p. 401

24. Let x be the distance *CD* lies from the centre, and let $5-x$ be the distance *AB* lies from the centre.

$$\therefore r^2 = x^2 + 3^2 \quad ①$$

$$r^2 = 2^2 + (5-x)^2 \quad ②$$



Substitute $x^2 + 3^2$ for r^2 into ②.

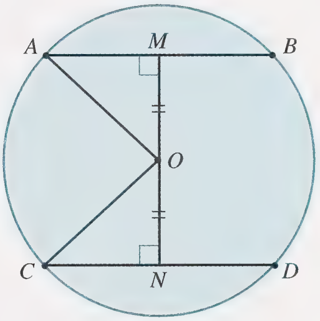
$$\begin{aligned} r^2 &= 2^2 + (5-x)^2 \\ x^2 + 3^2 &= 2^2 + (5-x)^2 \\ x^2 + 9 &= 4 + 25 - 10x + x^2 \\ 9 &= 29 - 10x \\ 10x &= 20 \\ x &= 2 \end{aligned}$$

Substitute $x = 2$ into ①.

$$r^2 = x^2 + 3^2$$
$$r^2 = 2^2 + 9$$
$$r^2 = 13$$
$$r = 3.6$$

The radius is about 3.6 cm.

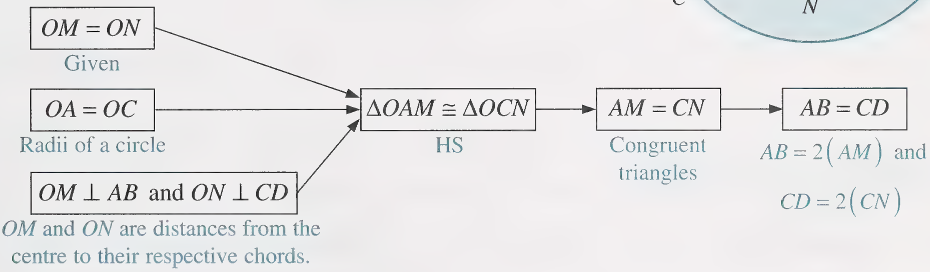
32. **Given:** Circle with centre O; chords AB and CD; $OM = ON$



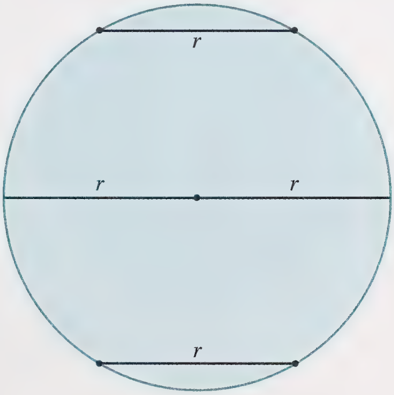
Prove: $AB = CD$

Proof:

In ΔOAM and ΔOCN

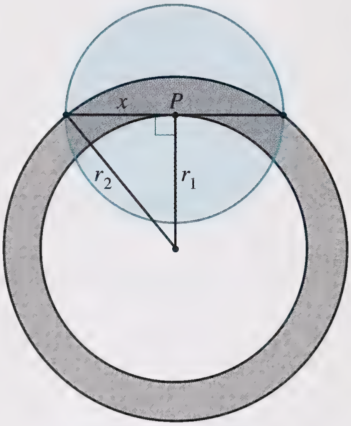

$OM = ON$
Given
$OA = OC$
Radii of a circle
$OM \perp AB$ and $ON \perp CD$
OM and ON are distances from the centre to their respective chords.
$\Delta OAM \cong \Delta OCN$
HS
$AM = CN$
Congruent triangles
$AB = CD$
$AB = 2(AM)$ and $CD = 2(CN)$

34. Only two chords of the same length as the radius can be drawn parallel to a given diameter.



Section 2: Activity 1 (continued)

5. Let r_1 be the radius of the inner circle; let r_2 be the radius of the outer circle; and let x be the radius of the circle with centre P.



$$\therefore \text{ Area of sidewalk} = \text{area of outer circle} - \text{area of inner circle}$$
$$= \pi r_2^{\ 2} - \pi r_1^{\ 2}$$
$$= \pi\left(r_2^{\ 2} - r_1^{\ 2}\right)$$

Now, find the radius of the circle with center P using the Pythagorean Theorem.

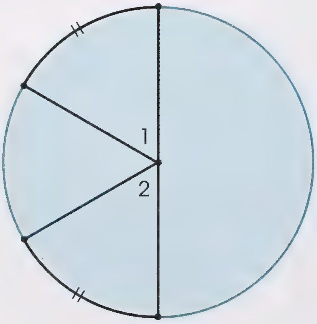
$$x^2 + r_1^{\ 2} = r_2^{\ 2}$$
$$x^2 = r_2^{\ 2} - r_1^{\ 2}$$

$$\therefore \text{ Area of circle} = \pi x^2$$
$$= \pi\left(r_2^{\ 2} - r_1^{\ 2}\right)$$
$$= \text{Area of sidewalk}$$

Section 2: Activity 2

1. a. **Textbook questions 1 to 7 of Investigation 1, "Central Angles Subtended by Equal Arcs," p.406**

   1. Drawings may vary. A sample drawing is given.



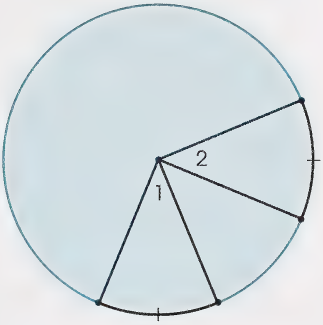
2. Refer to the answer to question 1.

3. Refer to the answer to question 1.

   $\angle 1 = 60°$ and $\angle 2 = 60°$

4. The measure of the angles are equal.

5. Drawings may vary. A sample drawing is given.



$\angle 1 = 45°$

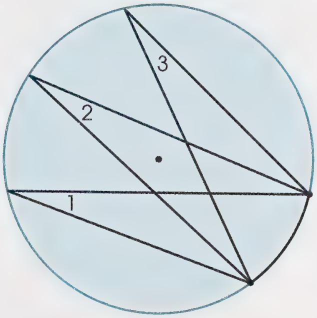
$\angle 2 = \angle 45°$

6. Your findings should agree with those of your classmates.

7. The measures of central angles subtended by equal arcs are equal.

b. Textbook questions 1 to 7 of Investigation 2, "Inscribed Angles Subtended by the Same Arc," p. 406

1. Drawings may vary. A sample drawing is given.



2. Refer to the answer to question 1.

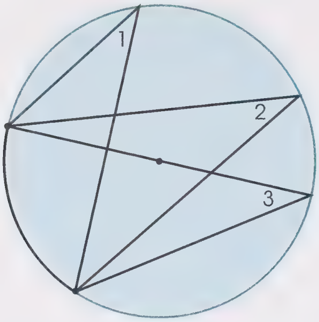
Section 2: Activity 2 (continued)

3. Refer to the answer to question 1.

$\angle 1 = 20°$, $\angle 2 = 20°$, and $\angle 3 = 20°$

4. The measures of the angles are equal.

5. Drawings may vary. A sample drawing is given.



$\angle 1 = 35°$

$\angle 2 = 35°$

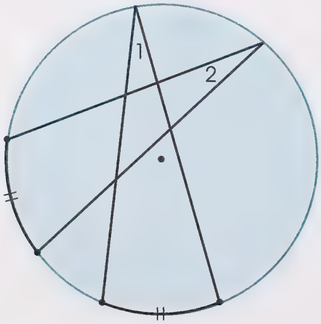
$\angle 3 = 35°$

6. Your findings should agree with those of your classmates.

7. The measures of inscribed angles subtended by the same arc are equal.

c. Textbook questions 1 to 7 of Investigation 3, "Inscribed Angles Subtended by Equal Arcs," p. 407

1. Drawings may vary. A sample drawing is given.



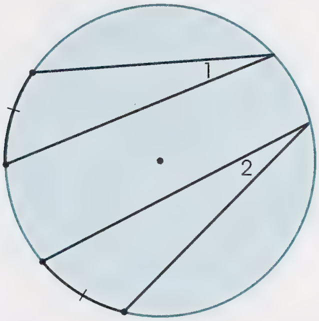
2. Refer to the answer to question 1.

3. Refer to the answer to question 1.

   $\angle 1 = 22°$ and $\angle 2 = 22°$

4. The measures of the angles are equal.

5. Drawings may vary. A sample drawing is given.



$\angle 1 = 18°$

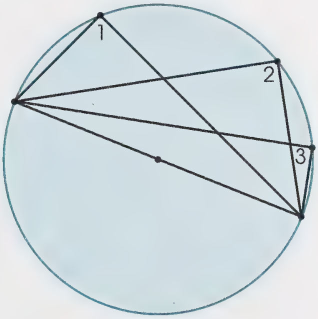
$\angle 2 = 18°$

6. Your results should agree with those of your classmates.

7. The measures of inscribed angles subtended by equal arcs are equal.

d. Textbook questions 1 to 7 of Investigation 4, "Angles Inscribed in a Semicircle," p. 407

1. Drawings may vary. A sample drawing is given.



2. Refer to the answer to question 1.

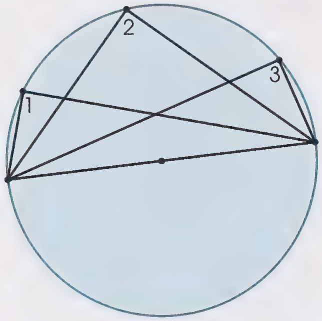
3. Refer to the answer to question 1.

   $\angle 1 = 90°$, $\angle 2 = 90°$, and $\angle 3 = 90°$

Section 2: Activity 2 (continued)

4. The measure of each angle is 90°.

5. Drawings may vary. A sample drawing is given.



$\angle 1 = 90°$

$\angle 2 = 90°$

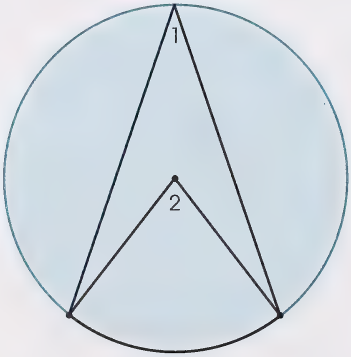
$\angle 3 = 90°$

6. Your results should agree with those of your classmates.

7. The measure of an angle inscribed in a semicircle is 90°.

e. Textbook questions 1 to 8 of Investigation 5, "Central and Inscribed Angles Subtended by the Same Arc," p. 407

1. Drawings may vary. A sample drawing is given.



2. Refer to the answer to question 1.

3. Refer to the answer to question 1.

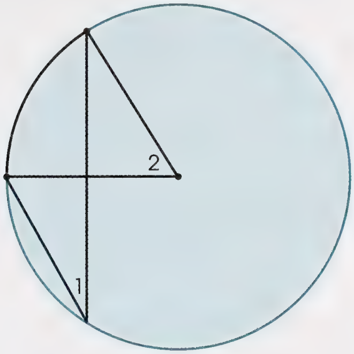
$\angle 2 = 76°$

4. Refer to the answer to question 1.

$\angle 1 = 38°$

5. The measure of the inscribed angle is one-half the measure of the central angle.

6. Drawings may vary. A sample drawing is given.



$\angle 1 = 29°$

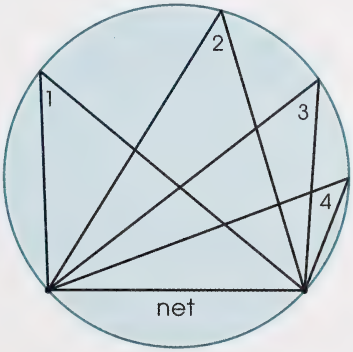
$\angle 2 = 58°$

7. Your findings should agree with those of your classmates.

8. The measure of the central angle is equal to twice the measure of the inscribed angle subtended by the same arc.

2. a. Textbook questions a. to c. of "Explore: Draw a Diagram," p. 408

a. Drawings may vary. A sample drawing is given.



b. Refer to the answer to question a.

c. $\angle 1 = \angle 2 = \angle 3 = \angle 4 = 48°$

Section 2: Activity 2 (continued)

b. Textbook questions 1 and 2 of "Inquire," p. 408

1. The four shooting angles are equal.

2. a. To increase the shooting angle, move the player inside the circle.
b. To decrease the shooting angle, move the player outside the circle.

3. a. Textbook questions 1, 3, 5, 6, 8, 9, 10, 13, 14, 19, 20, 22, and 24 of "Practice," pp. 412 and 413

1. $\angle BAC$ and $\angle BDC$

3. a. $\angle ADB$, $\angle ACB$, and $\angle AOB$ **b.** $\angle DAC$ and $\angle DBC$

5.
$$\begin{aligned} x &= \frac{1}{2}(80°) \\ &= 40° \end{aligned}$$

The measure of the inscribed angle is one-half the measure of the central angle.

6.
$$\begin{aligned} x &= 2(30°) \\ &= 60° \end{aligned}$$

The measure of the central angle is twice the measure of the inscribed angle.

8. $x = 90°$

An angle inscribed in a semicircle is 90°.

9.
$$\begin{aligned} x &= \frac{1}{2}(240°) \\ &= 120° \end{aligned}$$

The measure of the inscribed angle is one-half the measure of the central angle.

10.
$$\begin{aligned} x &= 2(43°) \\ &= 86° \end{aligned}$$

The measure of the central angle is twice the measure of the inscribed angle.

13. $a = 48°$, $b = 42°$, and $c = 48° + 42°$

$$= 90°$$

14.
$$\begin{aligned} x &= \frac{1}{2}(360° - 140°) \\ &= \frac{1}{2}(220°) \\ &= 110° \end{aligned}$$

19.

Measures	**Reasons**
$a = 56°$	Inscribed angles subtended by the same arc are equal.
$b = 2(56°)$ $= 112°$	The measure of the central angle is twice the measure of the inscribed angle.
$c = 180° - b$ $= 180° - 112°$ $= 68°$	$\angle b$ and $\angle c$ form a straight angle.
$d = 90° - 56°$ $= 34°$	An angle inscribed in a semicircle is 90°.

20.

Measures	**Reasons**
$x = 50°$	Inscribed angles subtended by the same arc are equal.
$z = 180° - (60° + x)$ $= 180° - (60° + 50°)$ $= 70°$	Triangle Angle Sum Theorem
$y = z$ $= 70°$	Inscribed angles subtended by the same arc are equal.
$t =$ angle adjacent to z $= 20°$	The angle adjacent to z was determined using the Triangle Angle Sum Theorem. For $\angle t$, inscribed angles subtended by the same arc are equal.

Section 2: Activity 2 (continued)

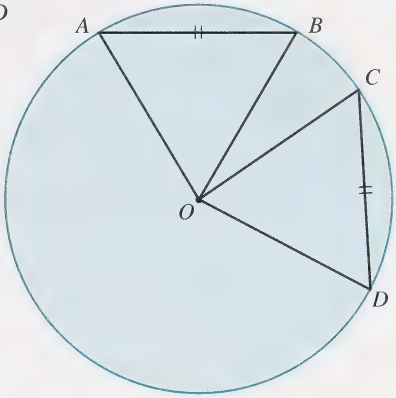
22.

Measures	**Reasons**
$x = z$	Isosceles Triangle Theorem
$x + z + 90^\circ = 180^\circ$ $x + x + 90^\circ = 180^\circ$ $2x = 80^\circ$ $x = 45^\circ$	An angle inscribed in a semicircle is 90°; Triangle Angle Sum Theorem
$y = 180^\circ - (90^\circ + 55^\circ)$ $= 180^\circ - 145^\circ$ $= 35^\circ$	An angle inscribed in a semicircle is 90°; Triangle Angle Sum Theorem
$z = x$ $= 45^\circ$	Proved

24.

Measures	**Reasons**
$y = 180^\circ - 140^\circ$ $= 40^\circ$	Supplementary angles
$x = 2y$ $= 2(40^\circ)$ $= 80^\circ$	The measure of the central angle is twice the measure of the inscribed angle.
$w = y$ $= 40^\circ$	Inscribed angles subtended by the same arc are equal.
$z + z + x = 180^\circ$ $2z + 80^\circ = 180^\circ$ $2z = 100^\circ$ $z = 50^\circ$	Triangle Angle Sum Theorem of an isosceles triangle

b. Textbook questions 29, 32.a., 35, and 38 of "Applications and Problem Solving," pp. 413 and 414



29. Given: Circle with centre O; chords AB and CD; $AB = CD$

Prove: $\angle AOB = \angle COD$

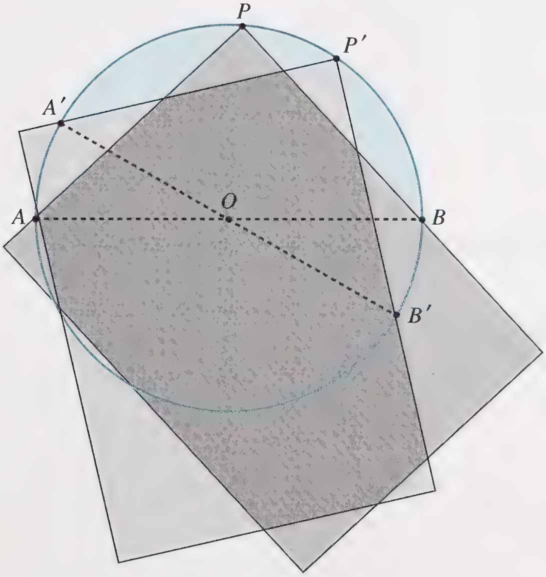
Proof:

Statements	**Reasons**
In ΔAOB and ΔCOD,	
$OA = OC$	Radii of a circle
$OB = OD$	Radii of a circle
$AB = CD$	Given
$\Delta AOB \cong \Delta COD$	SSS
$\angle AOB = \angle COD$	Congruent triangles

32. a. Put one corner of the sheet of paper on the circle at point P. Mark points A and B, and join them. AB is a diameter of the circle.

Draw a second diameter $A'B'$ using the same technique.

The diameters cross at the centre of the circle, O.



Section 2: Activity 2 (continued)

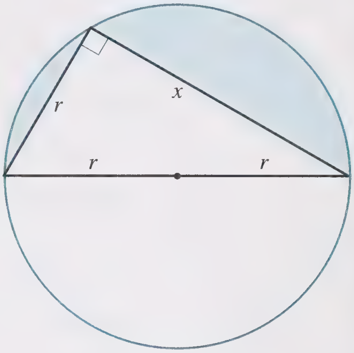
35. If $\angle XSY$ is less than 50°, the ship has not reached the circular boundary of the shallow and dangerous water.

Right at the boundary, $\angle XSY = 50°$.

If the ship enters the circular region, $\angle XSY > 50°$.

To assure the captain that he has not entered the dangerous region, the measure of $\angle XSY$ must be less than 50°.

38. Use the Pythagorean Theorem to express x in tems of r.



$$\begin{aligned} x^2 + r^2 &= (2r)^2 \\ x^2 + r^2 &= 4r^2 \\ x^2 &= 3r^2 \\ x &= \sqrt{3}r \end{aligned}$$

Determine the area of the right triangle.

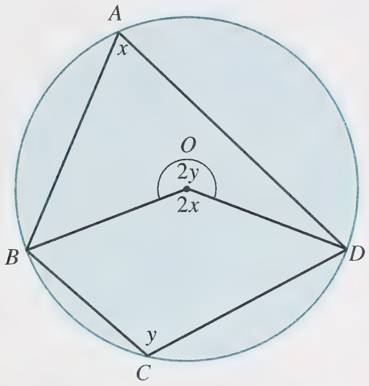
$$\begin{aligned} A &= \frac{1}{2}xr \\ &= \frac{1}{2}(\sqrt{3}r)(r) \\ &= \frac{\sqrt{3}}{2}r^2 \end{aligned}$$

The area of the semicircle with the diameter as its boundary is $\frac{\pi r^2}{2}$.

$$\begin{aligned} \therefore \text{ Area of shaded area} &= \frac{\pi r^2}{2} - \frac{\sqrt{3}}{2}r^2 \\ &= \frac{(\pi - \sqrt{3})r^2}{2} \end{aligned}$$

4. a. **Textbook questions a. and b. of "Explore: Use a Diagram," p. 415**

a.



Central $\angle BOD = 2\angle A$ ← Subtended by arc *BCD*

$= 2x$

b. Central $\angle BOD = 2\angle C$ ← Subtended by arc *BAD*

$= 2y$

b. **Textbook questions 1 to 7 of "Inquire," p. 415**

1. $2x + 2y$

2. $2x + 2y = 360°$

3. $x + y = 180°$

4. $\angle A + \angle C = 180°$

5. Because the sum of the angles of a quadrilateral is 360°,

$$\angle A + \angle C + \angle B + \angle D = 360°$$
$$180° + \angle B + \angle D = 360°$$
$$\angle B + \angle D = 180°$$

6. The sum of the opposite angles of a cyclic quadrilateral is 180°.

Section 2: Activity 2 (continued)

7. a.

Measures	**Reasons**
$x + 85^\circ = 180^\circ$ $x = 95^\circ$	Opposite angles of a cyclic quadrilateral
$y + 123^\circ = 180^\circ$ $y = 57^\circ$	Opposite angles of a cyclic quadrilateral

b.

Measures	**Reasons**
$a + 94^\circ = 180^\circ$ $a = 86^\circ$	Opposite angles of a cyclic quadrilateral
$b + 113^\circ = 180^\circ$ $= 67^\circ$	Opposite angles of a cyclic quadrilateral
$b + c = 180^\circ$ $67^\circ + c = 180^\circ$ $c = 113^\circ$	Supplementary angles

c.

Measures	**Reasons**
$d = 180^\circ - 127^\circ$ $= 53^\circ$	Opposite angles of a cyclic quadrilateral
$e = 180^\circ - d$ $= 180^\circ - 53^\circ$ $= 127^\circ$	Supplementary angles
$f + 76^\circ = 180^\circ$ $= 104^\circ$	Opposite angles in a cyclic quadrilateral
$f + g = 180^\circ$ $104^\circ + g = 180^\circ$ $g = 76^\circ$	Supplementary angles

5. a. **Textbook questions 1 and 3 to 8 of "Practice," pp. 419 and 420**

1. $\angle 1$ and 96° are opposite angles in a cyclic quadrilateral.

$$\therefore\ \angle 1 + 96^\circ = 180^\circ$$
$$\angle 1 = 84^\circ$$

$\angle 2$ and 107° are opposite angles in a cyclic quadrilateral.

$$\therefore\ \angle 2 + 107^\circ = 180^\circ$$
$$\angle 2 = 73^\circ$$

3. $\angle 1$ and 85° are interior angles on the same side of a transversal that intersects a pair of parallel lines.

$$\therefore\ \angle 1 + 85^\circ = 180$$
$$\angle 1 = 95^\circ$$

$\angle 2$ and 85° are opposite angles of a cyclic quadrilateral.

$$\therefore\ \angle 2 + 85^\circ = 180^\circ$$
$$\angle 2 = 95^\circ$$

$\angle 1$ and $\angle 3$ are opposite angles of a cyclic quadrilateral.

$$\therefore\ \angle 1 + \angle 3 = 180^\circ$$
$$95^\circ + \angle 3 = 180^\circ$$
$$\angle 3 = 85^\circ$$

4. $\angle 2$ and 110° are opposite angles of a cyclic quadrilateral.

$$\therefore\ \angle 2 + 110^\circ = 180^\circ$$
$$\angle 2 = 70^\circ$$

$\angle 3 = \angle 2$ because they lie opposite the congruent sides of an isosceles triangle.

$$\therefore\ \angle 3 = 70^\circ$$

Section 2: Activity 2 (continued)

Using the Triangle Angle Sum Theorem,

$$\begin{aligned}\angle 1+\angle 2+\angle 3&=180^\circ\\ \angle 1+70^\circ+70^\circ&=180^\circ\\ \angle 1&=40^\circ\end{aligned}$$

5. $\angle 1$ is an inscribed angle. It is one-half the measure of the central angle subtended by the same arc.

$$\begin{aligned}\therefore\ \angle 1&=\frac{1}{2}(86^\circ)\\ &=43^\circ\end{aligned}$$

$\angle 2$ is one of two equal angles in an isosceles triangle with a vertical angle of 86°.

$$\begin{aligned}\therefore\ 2(\angle 2)+86^\circ&=180^\circ\\ 2(\angle 2)&=94^\circ\\ \angle 2&=47^\circ\end{aligned}$$

$(\angle 2+\angle 3)$ and 107° are opposite angles of a cyclic quadrilateral.

$$\begin{aligned}\therefore\ \angle 2+\angle 3+107^\circ&=180^\circ\\ 47^\circ+\angle 3+107^\circ&=180^\circ\\ \angle 3+154^\circ&=180^\circ\\ \angle 3&=26^\circ\end{aligned}$$

6. $\angle 1$ is a central angle. It is twice the measure of the inscribed angle subtended by the same arc.

$$\begin{aligned}\therefore\ \angle 1&=2(80^\circ)\\ &=160^\circ\end{aligned}$$

$\angle 2$ and 80° are opposite angles of a cyclic quadrilateral.

$$\begin{aligned}\therefore\ \angle 2+80^\circ&=180^\circ\\ \angle 2&=100^\circ\end{aligned}$$

7. $\angle 1$ is a central angle. It is twice the measure of the inscribed angle subtended by the same arc.

$$\begin{aligned} \therefore\ \angle 1 &= 2(65^\circ) \\ &= 130^\circ \end{aligned}$$

$\angle 2$ and 65° are opposite angles in a cyclic quadrilateral.

$$\begin{aligned} \therefore\ \angle 2 + 65^\circ &= 180^\circ \\ \angle 2 &= 115^\circ \end{aligned}$$

Because a quadrilateral can be divided into two triangles, its angle sum is 360°.

$$\begin{aligned} \therefore\ \angle 1 + \angle 2 + \angle 3 + 55^\circ &= 360^\circ \\ 130^\circ + 115^\circ + \angle 3 + 55^\circ &= 360^\circ \\ \angle 3 + 300^\circ &= 360^\circ \\ \angle 3 &= 60^\circ \end{aligned}$$

8. The 106° angle and $\angle 1$ form a straight angle.

$$\begin{aligned} \therefore\ \angle 1 + 106^\circ &= 180^\circ \\ \angle 1 &= 74^\circ \end{aligned}$$

The 106° angle is an exterior angle. The interior opposite angle is the same measure.

$$\therefore\ \angle 2 = 106^\circ$$

$\angle 3$ and $\angle 4$ are congruent angles in an isosceles triangle.

$$\begin{aligned} \therefore\ \angle 3 + \angle 4 + \angle 2 &= 180^\circ \\ \angle 3 + \angle 4 + 106^\circ &= 180^\circ \\ \angle 3 + \angle 4 &= 74^\circ \end{aligned}$$

$$\therefore\ \angle 3 = \angle 4 = 37^\circ \quad \longleftarrow 74^\circ \div 2 = 37^\circ$$

$\angle 4$ and $\angle 5$ form a straight angle.

$$\begin{aligned} \therefore\ \angle 4 + \angle 5 &= 180^\circ \\ 37^\circ + \angle 5 &= 180^\circ \\ \angle 5 &= 143^\circ \end{aligned}$$

Section 2: Activity 2 (continued)

b. **Textbook questions 15 and 16 of "Applications and Problem Solving," p. 420**

15.

$$\begin{aligned}\angle B+\angle D&=180^\circ\\(2x+8)+(3x+2)&=180^\circ\\5x+10^\circ&=180^\circ\\5x&=170^\circ\\x&=34^\circ\end{aligned}$$

← Opposite angles of a cyclic quadrilateral

Substitute $x=34^\circ$ into $\angle C$.

$$\begin{aligned}\angle C&=3x-4\\&=3(34)-4\\&=98^\circ\end{aligned}$$

$$\begin{aligned}\therefore\ \angle A+\angle C&=180^\circ\\\angle A+98^\circ&=180^\circ\\\angle A&=82^\circ\end{aligned}$$

← Opposite angles of a cyclic quadrilateral

16.

$$\begin{aligned}\angle 1+94^\circ&=180^\circ\\\angle 1&=86^\circ\end{aligned}$$

← Opposite angles of a cyclic quadrilateral

$$\begin{aligned}\angle 1+\angle 5&=180^\circ\\86^\circ+\angle 5&=180^\circ\\\angle 5&=94^\circ\end{aligned}$$

← Supplementary angles

$$\begin{aligned}\angle 4+\angle 5+19^\circ&=180^\circ\\\angle 4+94^\circ+19^\circ&=180^\circ\\\angle 4+113^\circ&=180^\circ\\\angle 4&=67^\circ\end{aligned}$$

← Triangle Angle Sum Theorem

$$\begin{aligned}\angle 2&=\angle 4\\&=67^\circ\end{aligned}$$

← The exterior angle of a cyclic quadrilateral equals the interior opposite angle.

$$\begin{aligned}\angle 2+\angle 3&=180^\circ\\67^\circ+\angle 3&=180^\circ\\\angle 3&=113^\circ\end{aligned}$$

← Opposite angles of a cyclic quadrilateral

6. a. **Textbook questions 11 and 14 of "Practice," p 420**

11. Quadrilateral *CDFE* is a cyclic quadrilateral because $\angle CDF$ and $\angle CEF$ are right angles and, therefore, supplementary.

Quadrilateral *ABED* also forms a cyclic quadrilateral because $\angle ADF = \angle BEF$ and these angles lie on the same side of *AB*.

14. *CEGF* is cyclic because $\angle CEG$ and $\angle CFG$ are right angles and, therefore, supplementary.

ADGF is cyclic because $\angle ADG$ and $\angle AFG$ are right angles and, therefore, supplementary.

BDGE is cyclic because $\angle BDG$ and $\angle BEG$ are right angles and, therefore, supplementary.

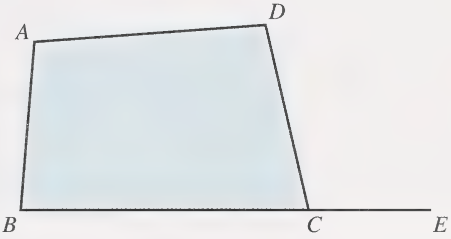
BDFC is cyclic because $\angle BDG = \angle CFG$ and these angles lie on the same side of *BC*.

ABEF is cyclic because $\angle BEG = \angle AFG$; these angles lie on the same side of *AB*.

ACED is cyclic because $\angle ADG = \angle CEG$; these angles lie on the same side of *AC*.

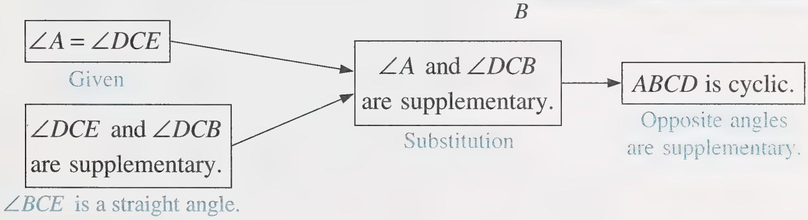
b. **Textbook questions 17 and 23 of "Applications and Problem Solving," pp. 420 and 421**

17. **Given:** Quadrilateral *ABCD* with exterior $\angle DCE$;
$\angle A = \angle DCE$



Prove: *ABCD* is cyclic.

Proof:



$\angle A = \angle DCE$
Given

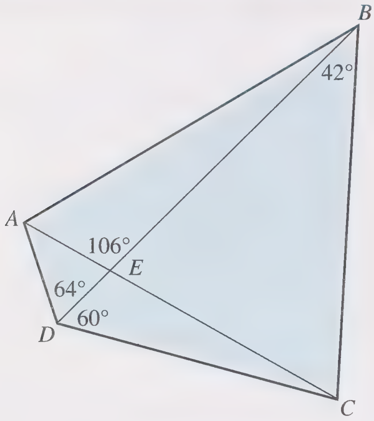
$\angle DCE$ and $\angle DCB$ are supplementary.
$\angle BCE$ is a straight angle.

$\angle A$ and $\angle DCB$ are supplementary.
Substitution

ABCD is cyclic.
Opposite angles are supplementary.

Section 2: Activity 2 (continued)

23. **a.**



Measures	**Reasons**
$\angle BEC + 106° = 180°$ $\angle BEC = 74°$	$\angle BEC$ and $\angle AEB$ are supplementary; $\angle AEC$ is a straight angle.
$\angle BCE + 74° + 42° = 180°$ $\angle BCE = 64°$	Triangle Angle Sum Theorem
ABCD is a cyclic quadrilateral.	$\angle ADE = \angle BCE$; Converse of the Cyclic Quadrilateral Theorem

b. Because opposite angles of a cyclic quadrilateral are supplementary,

$$\begin{aligned} \angle ABC + \angle ADC &= 180° \\ (\angle DBA + 42°) + (64° + 60°) &= 180° \\ \angle DBA + 42° + 124° &= 180° \\ \angle DBA + 166° &= 180° \\ \angle DBA &= 14° \end{aligned}$$

7. Textbook questions 1 and 2 of "PATTERN POWER," p. 421

1. In each row, the product of the first pair of numbers is one-half the product of the second pair.

2. Let x be the missing number.

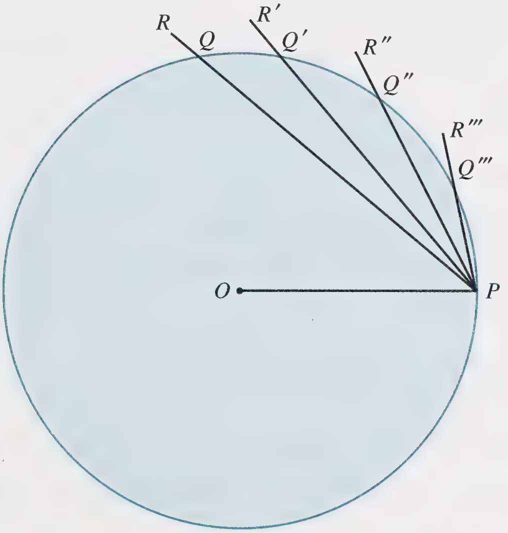
$$\begin{aligned}\therefore\ 10\times3&=\frac{1}{2}(x\times4)\\ 30&=2x\\ 2x&=30\\ x&=15\end{aligned}$$

The missing number is 15.

Section 2: Activity 3

1. a. Textbook questions a. to c. of Investigation 1 of "Explore: Use the Diagrams," p. 426

a. to c.



$\angle OPR = 40°$

$\angle OPR' = 50°$

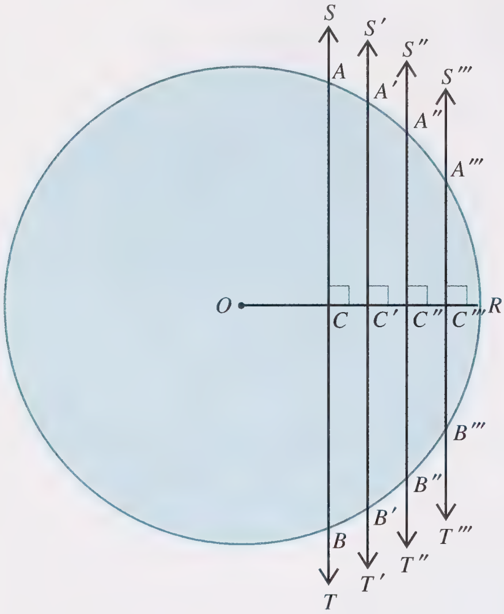
$\angle OPR'' = 63°$

$\angle OPR''' = 78°$

Section 2: Activity 3 (continued)

b. Textbook questions a. to c. of Investigation 2 of "Explore: Use the Diagrams," p. 426

a. to c.



c. Textbook questions 1 to 3 of "Inquire," p. 427

1. **a.** As Q moves closer to P, the length of PQ approaches 0.
   b. As Q moves closer to P, the measure of $\angle OPR$ approaches 90°.
   c. When Q touches P, $\angle OPR = 90°$.

2. **a.** As C moves closer to R, the length of AB approaches 0.
   b. When C touches R, $AB = 0$.
   c. When C touches R, line ST is tangent to the circle at R.

3. The angle between a tangent to a circle and a radius drawn to the point of tangency is 90°.

2. a. Textbook questions 1 to 6 of "Practice," p. 431

1. Because tangent AP is perpendicular to radius OA, ΔOAP is a right triangle.

Use the Pythagorean Theorem to find x.

$$\begin{aligned} x^2 + 12^2 &= 13^2 \\ x^2 + 144 &= 169 \\ x^2 &= 25 \\ x &= 5 \end{aligned}$$

2. Because tangent segments from an external point are congruent, $PB = PA$ and $y = 9$.

By symmetry, $\Delta APO \cong \Delta BPO$. Therefore, $\angle POA = \angle POB$ and $\angle x = 70°$.

Also, $\angle w + 70° + 90° = 180°$

$$\angle w = 20°$$

Because $\Delta APO \cong \Delta BPO$ and $\angle w = \angle z$, $\angle z = 20°$.

3. $$\begin{aligned} x^2 &= 5^2 + 10^2 \\ x^2 &= 25 + 100 \\ x^2 &= 125 \\ x &= \sqrt{125} \\ &= 5\sqrt{5} \end{aligned}$$

4. Because tangent segments of a circle with centre O from an external point are congruent, $x = 13$.

Because x and y are tangent segments drawn to the circle with centre M, $x = y = 13$.

Section 2: Activity 3 (continued)

5.

$$
\begin{aligned}
(x+5)^2 &= 5^2+9^2\\
x^2+10x+25 &= 25+81\\
x^2+10x-81 &= 0
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
&= \frac{-10\pm\sqrt{10^2-4(1)(-81)}}{2(1)}\\
&= \frac{-10\pm\sqrt{100+324}}{2}\\
&= \frac{-10\pm\sqrt{424}}{2}\\
&= \frac{-10\pm 2\sqrt{106}}{2}\\
&= \frac{\cancel{2}\left(-5\pm\sqrt{106}\right)}{\cancel{2}}\\
&= -5\pm\sqrt{106}
\end{aligned}
$$

Note:

$a=1$

$b=10$

$c=-81$

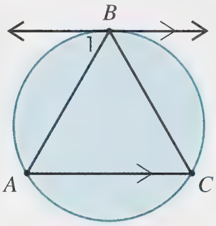
Because x is non-negative, $x=-5+\sqrt{106}$.

6.

$$
\begin{aligned}
\angle A+\angle B+x+40^\circ &= 360^\circ\\
90^\circ+90^\circ+x+40^\circ &= 360^\circ\\
x &= 140^\circ
\end{aligned}
$$

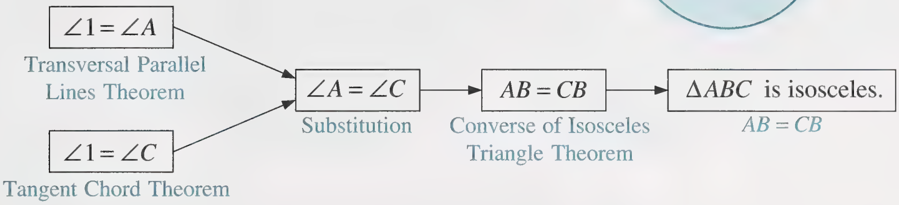
b. **Textbook questions 15, 18, 19, 20, 23, 27, and 31 of "Applications and Problem Solving," pp. 431 and 432**

15. **Given:** Tangent at B and parallel to AC



Prove: ΔABC is isosceles.

Proof:



$\angle 1 = \angle A$

Transversal Parallel Lines Theorem

$\angle 1 = \angle C$

Tangent Chord Theorem

$\angle A = \angle C$

Substitution

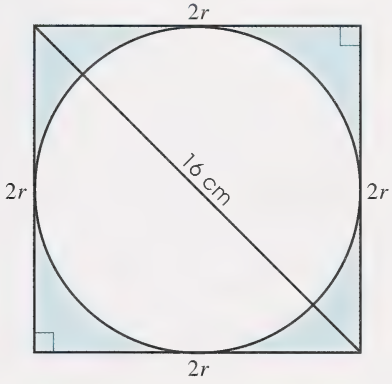
$AB = CB$

Converse of Isosceles Triangle Theorem

ΔABC is isosceles.

$AB = CB$

18. **a.** Let r be the radius of the circle.



$$
\begin{aligned}
\therefore\ (2r)^2 + (2r)^2 &= 16^2 \\
4r^2 + 4r^2 &= 256 \\
8r^2 &= 256 \\
r^2 &= 32 \\
r &= \sqrt{32} \\
&= 4\sqrt{2} \\
&\doteq 5.7
\end{aligned}
$$

The radius of the circle is about 5.7 cm.

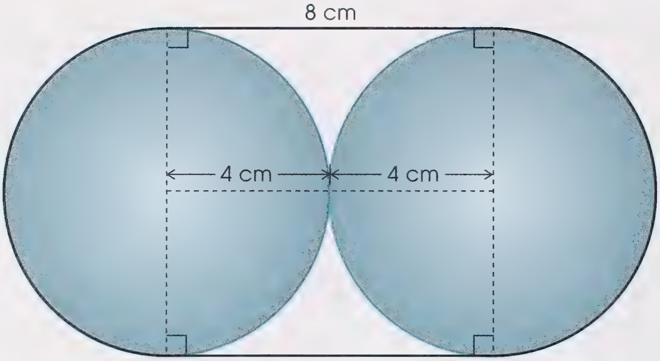
b. Area of piece left over = area of square − area of circle

$$
\begin{aligned}
&= (2r)^2 - \pi r^2 \\
&= 4r^2 - \pi r^2 \\
&= 4(32) - \pi(32) \\
&= 128 - 32\pi \\
&\doteq 27.5
\end{aligned}
$$

The area of the piece left over is approximately 27.5 cm^2.

Section 2: Activity 3 (continued)

19.



$d = 8$ cm

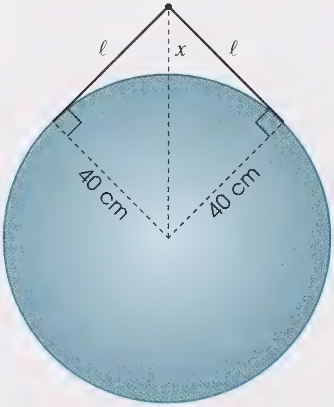
$\therefore\ r = 4$ cm

$$\begin{aligned}\text{The total length of the band} &= 2(8) + 2 \text{ semicircles}\\ &= 16 + 2(\pi r)\\ &= 16 + 2\pi(4)\\ &\doteq 41.1\end{aligned}$$

The length of the band is about 41.1 cm.

20. $2\ell = 60$

$\ell = 30$



Let x be the height of the hook above the gong.

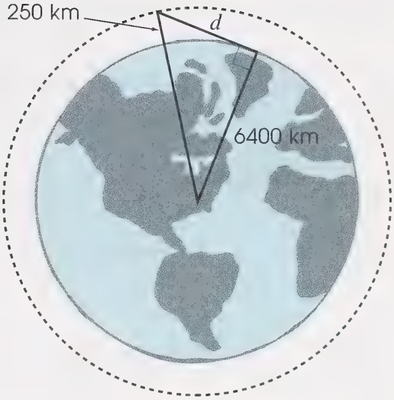
$$\begin{aligned}\therefore\ \ell^2 + 40^2 &= (x+40)^2\\ (30)^2 + 1600 &= x^2 + 80x + 1600\\ 900 + 1600 &= x^2 + 80x + 1600\\ 0 &= x^2 + 80x - 900\\ 0 &= (x+90)(x-10)\end{aligned}$$

$\therefore\ x + 90 = 0$ or $x - 10 = 0$

$x = -90$ $\quad$ $x = 10$

Because x is non-negative, the hook is 10 cm above the gong.

23. Let d be the distance from the shuttle to the horizon.



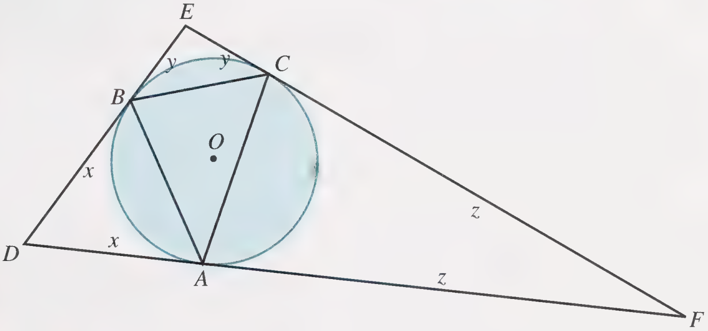
$$\therefore\ d^2 + 6400^2 = (6400 + 250)^2$$
$$d^2 = 6650^2 - 6400^2$$
$$d^2 = 3\,262\,500$$
$$d \doteq 1806\ \text{km}$$

The distance from the shuttle to the horizon is about 1806 km.

27. $DE = 52 = x + y$ ①

$DF = 89 = x + z$ ②

$EF = 74 = y + z$ ③



$52 = x + y$ ①

$89 = x \quad + z$ ②

$-37 = \quad y - z$ ④ : ① − ②

$74 = y + z$ ③

$-37 = y - z$ ④

$37 = 2y$ ③ + ④

$18.5 = y$

Substitute $y = 18.5$ into ① and ③.

$52 = x + y$	$74 = y + z$
$52 = x + 18.5$	$74 = 18.5 + z$
$33.5 = x$	$55.5 = z$

$\therefore\ AD = x = 33.5$

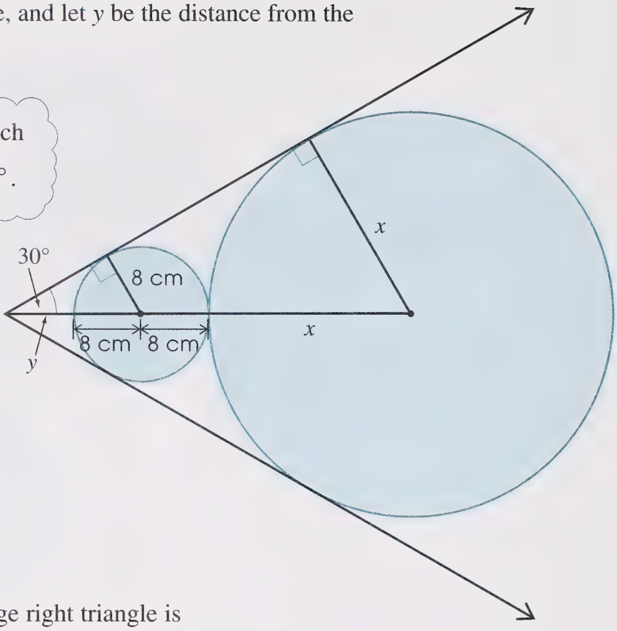
$AF = z = 55.5$

$CE = y = 18.5$

Section 2: Activity 3 (continued)

31. Let x be the radius of the large circle, and let y be the distance from the 60° angle to the smaller circle.

The smaller acute angle in each right triangle is $\frac{1}{2} \times 60° = 30°$.



First, solve for y.

$$\sin 30° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\frac{1}{2} = \frac{8}{y+8}$$

$$y + 8 = 16$$

$$y = 8$$

Therefore, the hypotenuse of the large right triangle is

$$\begin{aligned} y + 8 + 8 + x &= 8 + 8 + 8 + x \\ &= x + 24 \end{aligned}$$

Now, solve for x.

$$\sin 30° = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\frac{1}{2} = \frac{x}{x+24}$$

$$x + 24 = 2x$$

$$24 = x$$

$$x = 24$$

The radius of the larger circle is 24 cm.

3. a. Textbook questions 7, 9, 11, and 13 of "Practice," p. 431

7. Using the Tangent Chord Theorem, $\angle 1 = 64°$. Similarly, using the Tangent Chord Theorem, $\angle 2 = 71°$.

9. To find the measure of $\angle 1$, calculate $\angle CDF$ first. Because $\angle EDF$ is a straight angle,

$$\begin{aligned} 45° + 105° + \angle CDF &= 180° \\ \angle CDF + 150° &= 180° \\ \angle CDF &= 30° \end{aligned}$$

Therefore, using the Tangent Chord Theorem, $\angle 1 = 30°$.

Because opposite angles of a cyclic quadrilateral are supplementary,

$$\begin{aligned} \angle 2 + 105° &= 180° \\ \angle 2 &= 75° \end{aligned}$$

11. In ΔEJK, $\angle EKJ = \angle 1$ because tangent segments EJ and EK are equal.

$$\begin{aligned} \therefore\ 2(\angle 1) + 82° &= 180° \quad \longleftarrow \text{Triangle Angle Sum Theorem} \\ 2(\angle 1) &= 98° \\ \angle 1 &= 49° \end{aligned}$$

Similarly, in ΔFKL, $\angle FKL = \angle 2$ because tangent segments FL and FK are equal.

$$\begin{aligned} \therefore\ 2(\angle 2) + 58° &= 180° \quad \longleftarrow \text{Triangle Angle Sum Theorem} \\ 2(\angle 2) &= 122° \\ \angle 2 &= 61° \end{aligned}$$

Because $\angle EKF$ is a straight angle,

$$\begin{aligned} \angle EKJ + \angle 3 + \angle FKL &= 180° \\ 49° + \angle 3 + 61° &= 180° \\ \angle 3 + 110° &= 180° \\ \angle 3 &= 70° \end{aligned}$$

Recall that $\angle EKJ = \angle 1$ and $\angle FKL = \angle 2$.

$$\begin{aligned} \angle 4 &= \angle 2 \quad \longleftarrow \text{Tangent Chord Theorem} \\ &= 61° \end{aligned}$$

Section 2: Activity 3 (continued)

In ΔDEF,

$$\begin{aligned}\angle 5+\angle E+\angle F&=180^\circ\\ \angle 5+82^\circ+58^\circ&=180^\circ\\ \angle 5+140^\circ&=180^\circ\\ \angle 5&=40^\circ\end{aligned}$$

← Triangle Angle Sum Theorem

13. In ΔABE,

$$\begin{aligned}\angle 1+30^\circ+73^\circ&=180^\circ\\ \angle 1+103^\circ&=180^\circ\\ \angle 1&=77^\circ\end{aligned}$$

← Triangle Angle Sum Theorem

$$\begin{aligned}\angle 2&=\angle DAE\\ &=37^\circ\end{aligned}$$

← Tangent Chord Theorem

$$\begin{aligned}\angle 3&=\angle BEA\\ &=73^\circ\end{aligned}$$

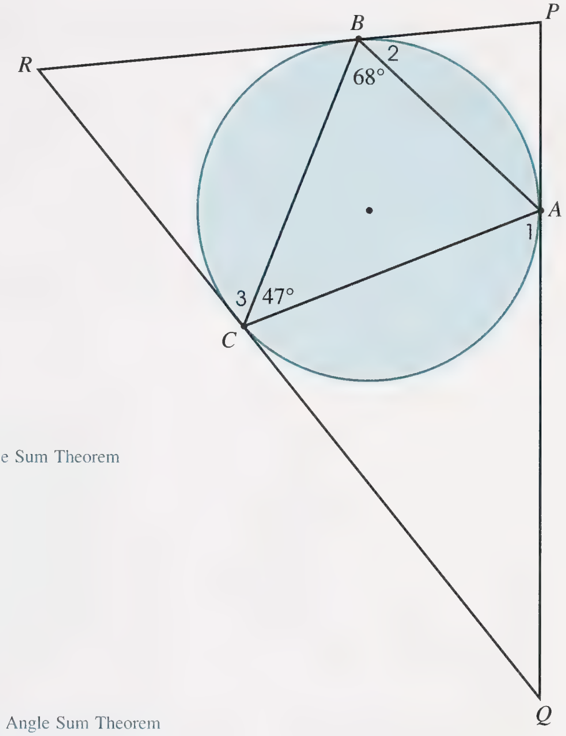
← Tangent Chord Theorem

In ΔCDE, $\angle 2=\angle DEC$.

$$\begin{aligned}\therefore\ 2(\angle 2)+\angle 4&=180^\circ\\ 2(37^\circ)+\angle 4&=180^\circ\\ \angle 4+74^\circ&=180^\circ\\ \angle 4&=106^\circ\end{aligned}$$

← Triangle Angle Sum Theorem

b. Textbook questions 16 and 26 of "Applications and Problem Solving," pp. 431 and 432



16.
$$\begin{aligned}\angle BAC + 68^\circ + 47^\circ &= 180^\circ\\ \angle BAC + 115^\circ &= 180^\circ\\ \angle BAC &= 65^\circ\end{aligned}$$

Using the Tangent Chord Theorem,

$$\angle 1 = 68^\circ$$
$$\angle 2 = 47^\circ$$
$$\angle 3 = 65^\circ$$

In ΔPAB, $\angle 2 = \angle PAB$.

$$\begin{aligned}\therefore\ \angle P + 2(\angle 2) &= 180^\circ \quad \longleftarrow \text{Triangle Angle Sum Theorem}\\ \angle P + 2(47^\circ) &= 180^\circ\\ \angle P + 94^\circ &= 180^\circ\\ \angle P &= 86^\circ\end{aligned}$$

Similarly, in ΔQAC, $\angle 1 = \angle QCA$.

$$\begin{aligned}\therefore\ \angle Q + 2(\angle 1) &= 180^\circ \quad \longleftarrow \text{Triangle Angle Sum Theorem}\\ \angle Q + 2(68^\circ) &= 180^\circ\\ \angle Q + 136^\circ &= 180^\circ\\ \angle Q &= 44^\circ\end{aligned}$$

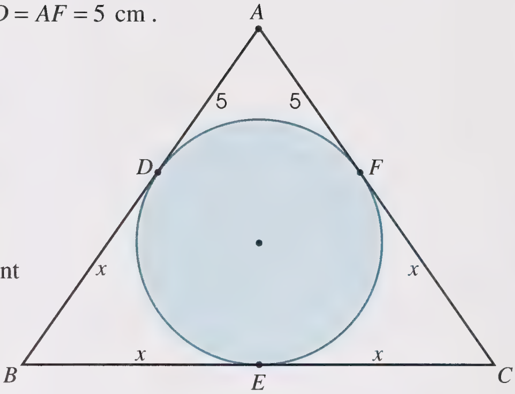
In ΔPQR,

$$\begin{aligned}\angle P + \angle Q + \angle R &= 180^\circ \quad \longleftarrow \text{Triangle Angle Sum Theorem}\\ 86^\circ + 44^\circ + \angle R &= 180^\circ\\ \angle R + 130^\circ &= 180^\circ\\ \angle R &= 50^\circ\end{aligned}$$

The angles in ΔPQR are $\angle P = 86^\circ$, $\angle Q = 44^\circ$, and $\angle R = 50^\circ$.

Section 2: Activity 3 (continued)

26. Because AD and AF are tangent segments, $AD = AF = 5\text{ cm}$.



Let x be DB.

$\therefore\ AB = AC = x + 5$

Because $AB = AC$, $DB = FC = x$.

Because DB and BE and FC and CE are tangent segments,

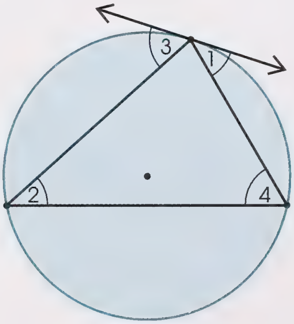
$$BE = DB = x$$
$$CE = FC = x$$

$$\begin{aligned}\therefore\ \text{Perimeter} &= 4x + 10\\ 62 &= 4x + 10\\ 4x + 10 &= 62\\ 4x &= 52\\ x &= 13\end{aligned}$$

$$\therefore\ AB = AC = x + 5 = 13 + 5 = 18\text{ cm} \quad \text{and} \quad BC = 2x = 2(13) = 26\text{ cm}$$

4. Textbook questions 1 to 8 of Investigation 4, "Angle Between a Tangent and a Chord," p. 425

1. Drawings may vary. A sample drawing is given.



2. Refer to the answer to question 1.

3. Refer to the answer to question 1.

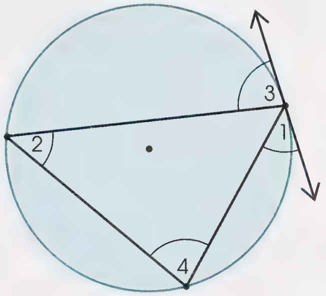
$\angle 1 = 42°$ and $\angle 3 = 60°$

4. Refer to the answer to question 1.

$\angle 2 = 42°$ and $\angle 4 = 60°$

5. The measure of the tangent chord angle is equal to the measure of the inscribed angle on the opposite side of the chord.

6. Drawings may vary. A sample drawing is given.



$\angle 1 = 46°$

$\angle 2 = 46°$

$\angle 3 = 79°$

$\angle 4 = 79°$

7. Your results should agree with those of your classmates.

8. The measure of a tangent chord angle is equal to the measure of an inscribed angle drawn on the opposite side of the chord.

Section 2: Activity 4

1. a. Textbook questions a. to d. of "Explore: Interpret the Information," p. 433

a. Area of field $= \pi r^2$

$$\begin{aligned} &= \pi(400)^2 \\ &= 160\,000\pi \text{ m}^2 \\ &= \frac{160\,000\pi \text{ m}^2}{10\,000 \text{ m}^2/\text{ha}} \\ &\doteq 50 \text{ ha} \end{aligned}$$

Note:

$10\,000\, m^2 = 1\, ha$

Section 2: Activity 4 (continued)

b. If the central angle is 90°, then $\frac{1}{4}$ of the field is irrigated.

$$\begin{aligned} A &\doteq \frac{1}{4}(50) \\ &\doteq 13\ \text{ha} \end{aligned}$$

If the central angle is 180°, then $\frac{1}{2}$ of the field is irrigated.

$$\begin{aligned} A &\doteq \frac{1}{2}(50) \\ &\doteq 25\ \text{ha} \end{aligned}$$

If the central angle is 270°, then $\frac{3}{4}$ of the field is irrigated.

$$\begin{aligned} A &\doteq \frac{3}{4}(50) \\ &\doteq 38\ \text{ha} \end{aligned}$$

c. $$\begin{aligned} \text{Circumference} &= 2\pi r \\ &= 2\pi(400) \\ &\doteq 2510\ \text{m} \end{aligned}$$

d. For a central angle of 90°,

$$\begin{aligned} \text{Arc length} &\doteq \frac{90°}{360°} \times 2510 \\ &\doteq 630\ \text{m} \end{aligned}$$

For a central angle of 45°,

$$\begin{aligned} \text{Arc length} &\doteq \frac{45°}{360°} \times 2510 \\ &\doteq 310\ \text{m} \end{aligned}$$

For a central angle of 135°,

$$\begin{aligned} \text{Arc length} &\doteq \frac{135°}{360°} \times 2510 \\ &\doteq 940\ \text{m} \end{aligned}$$

b. Textbook questions 1 to 5 of "Inquire," p. 433

1. **a.** A central angle of 60° is $\frac{60^\circ}{360^\circ}$ or $\frac{1}{6}$ of the area of the circle.

   b. A central angle of 30° is $\frac{30^\circ}{360^\circ}$ or $\frac{1}{12}$ of the area of the circle.

   c. A central angle of 120° is $\frac{120^\circ}{360^\circ}$ or $\frac{1}{3}$ of the area of the circle.

2. $A = \dfrac{m^\circ}{360^\circ} \times \pi r^2$

3. **a.** A central angle of 20° is $\frac{20^\circ}{360^\circ}$ or $\frac{1}{18}$ of the circumference.

   b. A central angle of 100° is $\frac{100^\circ}{360^\circ}$ or $\frac{5}{18}$ of the circumference.

   c. A central angle of 300° is $\frac{300^\circ}{360^\circ}$ or $\frac{5}{6}$ of the circumference.

4. $\ell = \dfrac{m^\circ}{360^\circ} \times 2\pi r$

5. **a.** $A = \dfrac{40^\circ}{360^\circ} \times \pi(10)^2 \doteq 35 \text{ cm}^2$ $\qquad \ell = \dfrac{40^\circ}{360^\circ} \times 2\pi(10) \doteq 7 \text{ cm}$

   b. $A = \dfrac{160^\circ}{360^\circ} \times \pi(15)^2 \doteq 314 \text{ cm}^2$ $\qquad \ell = \dfrac{160^\circ}{360^\circ} \times 2\pi(15) \doteq 42 \text{ cm}$

   c. $A = \dfrac{225^\circ}{360^\circ} \times \pi(50)^2 \doteq 4909 \text{ cm}^2$ $\qquad \ell = \dfrac{225^\circ}{360^\circ} \times 2\pi(50) \doteq 196 \text{ cm}$

   d. $A = \dfrac{300^\circ}{360^\circ} \times \pi(40)^2 \doteq 4189 \text{ cm}^2$ $\qquad \ell = \dfrac{300^\circ}{360^\circ} \times 2\pi(40) \doteq 209 \text{ cm}$

Section 2: Activity 4 (continued)

2. a. Textbook questions 2, 7, 9, 15, and 27 of "Practice," pp. 435 and 436

2. $m = 121°$ and $r = 30$ cm

$$\begin{aligned} \therefore\ \ell &= \frac{m}{360°} \times 2\pi r \\ &= \frac{121°}{360°} \times 2\pi(30) \\ &\doteq 63.4 \text{ cm} \end{aligned}$$

7. $m = 275°$ and $r = 55$ cm

$$\begin{aligned} \therefore\ A &= \frac{m}{360°} \times \pi r^2 \\ &= \frac{275°}{360°} \times \pi(55)^2 \\ &= 7259.5 \text{ cm}^2 \end{aligned}$$

9. $m = a$, $r = 2.5$ cm, and $\ell = 3.8$ cm

$$\begin{aligned} \therefore\ \ell &= \frac{m}{360°} \times 2\pi r \\ 3.8 &= \frac{a}{360°} \times 2\pi(2.5) \\ a &= \frac{360°(3.8)}{2\pi(2.5)} \\ &\doteq 87° \end{aligned}$$

15. $m = 125°$ and $r = 15$ cm

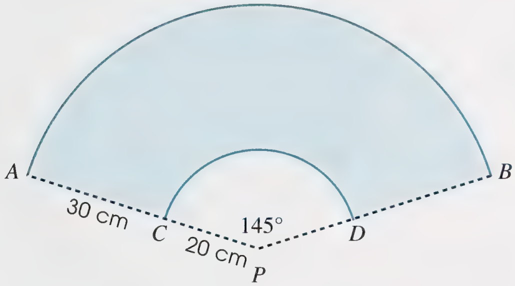
$$\begin{aligned} \therefore\ \ell &= \frac{m°}{360°} \times 2\pi r \\ &= \frac{125°}{360°} \times 2\pi(15) \\ &\doteq 32.7 \text{ cm} \end{aligned}$$

27. $m = 70°$ and $r = 5$ cm

$$\begin{aligned}\therefore\ A &= \frac{m}{360°} \times \pi r^2 \\ &= \frac{70°}{360°} \times \pi(5)^2 \\ &\doteq 15.3 \text{ cm}^2\end{aligned}$$

b. Textbook questions 34, 36, and 42 of "Applications and Problem Solving," pp. 436 and 437

34.



$$\begin{aligned}\text{Area of sector } ABP &= \frac{145°}{360°} \times \pi(50)^2 \\ &\doteq 3163.4\end{aligned}$$

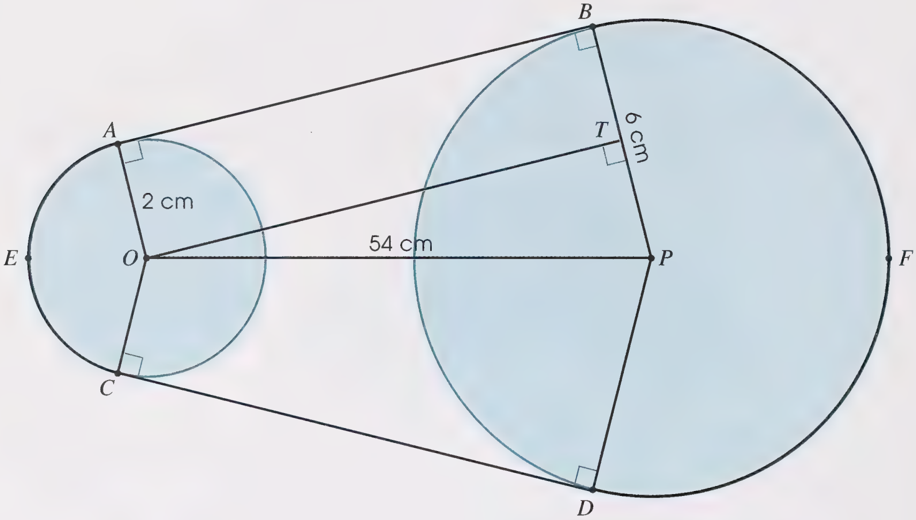
$$\begin{aligned}\text{Area of sector } CDP &= \frac{145°}{360°} \times \pi(20)^2 \\ &\doteq 506.1\end{aligned}$$

$$\begin{aligned}\text{Area cleaned by blade} &= \text{area of sector } ABP - \text{area of sector } CDP \\ &\doteq 3163.4 - 506.1 \\ &\doteq 2657.3\end{aligned}$$

The area cleaned by the windshield wiper is approximately 2657 cm^2.

Section 2: Activity 4 (continued)

36. Draw a diagram. Label the centres *O* and *P*; the points of tangency *A*, *B*, *C*, and *D*; and points *E* and *F* on the circles (as shown). Draw $OT \perp PB$ (as shown).



To find the minimum length of chain, find the sum of tangents *AB* and *CD* and the lengths of arcs *AEC* and *BFD*.

First, find *AB* and *CD*.

Because $AB = OT$, you can find *AB* using ΔOTP.

In ΔOTP, $OP = 54$ cm and $PT = 6 - 2 = 4$ cm.

$$\begin{aligned} \therefore\ (OT)^2 + (PT)^2 &= (OP)^2 \\ (OT)^2 &= (OP)^2 - (PT)^2 \\ (OT)^2 &= 54^2 - 4^2 \\ (OT)^2 &= 2900 \\ OT &\doteq 53.85 \end{aligned}$$

Therefore, $AB \doteq 53.85$ cm.

Because $AB = CD$, $CD \doteq 53.85$ cm.

Before you can determine the length of arcs *AEC* and *BED*, you must find the measures of $\angle BPD$ and $\angle AOC$.

Because $\angle BPD = 2(\angle OPT)$, calculate $\angle OPT$.

$$\begin{aligned}\cos\angle OPT &= \frac{4}{54}\\ \angle OPT &\doteq 85.75^\circ\end{aligned}$$

$$\begin{aligned}\therefore\ \angle BPD &= 2(\angle OPT)\\ &\doteq 2(85.75^\circ)\\ &\doteq 171.50^\circ\end{aligned}$$

Because $\angle BPD$ and $\angle AOC$ are formed by the intersection of the same two sets of parallel lines, $\angle BPD = \angle AOC$.

$\therefore\ \angle AOC = 171.50^\circ$

Determine the length of arc *AEC*.

$$\begin{aligned}\text{Arc } AEC &\doteq \frac{171.50^\circ}{360^\circ}\times 2\pi(2)\\ &\doteq 5.99\end{aligned}$$

Determine the length of arc *BFD*. You must use reflex angle *BPD*.

$$\begin{aligned}\text{Reflex angle } BPD &\doteq 360^\circ - 171.50^\circ\\ &\doteq 188.50^\circ\end{aligned}$$

$$\begin{aligned}\text{Arc } BFD &\doteq \frac{188.50^\circ}{360^\circ}\times 2\pi(6)\\ &\doteq 19.73\end{aligned}$$

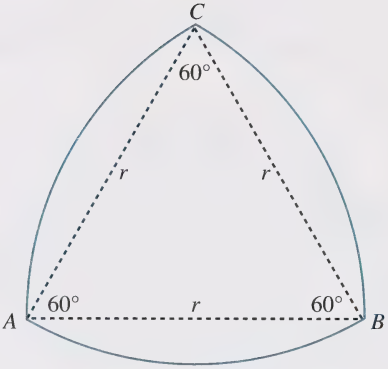
$$\begin{aligned}\therefore\ \text{Total length} &= AB + CD + \text{arc } AEC + \text{arc } BFD\\ &\doteq 53.85 + 53.85 + 5.99 + 19.73\\ &\doteq 133\end{aligned}$$

The minimum length of chain needed is approximately 133 cm.

Section 2: Activity 4 (continued)

42. Arc AB = Arc BC = Arc AC

$$\begin{aligned} \therefore \text{ Perimeter} &= 3\times \text{Arc } AB \\ &= 3\times\frac{60^\circ}{360^\circ}\times 2\pi r \\ &= 3\times\frac{1}{6}\times 2\pi r \\ &= \frac{6\pi r}{6} \\ &= \pi r \end{aligned}$$



An expression for the perimeter of the rotor is πr.

3. a. Textbook question "Explore: Look for a Pattern," p. 439

Polygon	Number of Sides	Number of Triangles	Sum of Interior Angle Measures (degrees)	Sum of Interior Angle Measures (in Right Angles)
triangle	3	1	180	2
quadrilateral	4	2	360	4
pentagon	5	3	540	6
hexagon	6	4	720	8
heptagon	7	5	900	10
octagon	8	6	1080	12
nonagon	9	7	1260	14
decagon	10	8	1440	16

b. Textbook questions 1 to 4 of "Inquire," p. 440

1. There are two fewer triangles formed by drawing diagonals from one vertex than the number of sides of the polygon.

2. **a.** There are 18 triangles formed.
   b. The sum of the measures of the interior angles is $18\times 180^\circ = 3240^\circ$, or 36 right angles.

3. **a.** There are $n-2$ triangles formed.
   b. The sum of the measures of the interior angle (in degrees) is $180°(n-2)$.
   c. The sum of the measures of the interior angles (in right angles) is $2(n-2)=2n-4$.

4. **a.**

$$\begin{aligned}\frac{180°(n-2)}{n}&=\frac{180°(8-2)}{8}\\&=\frac{180°(6)}{8}\\&=135°\end{aligned}$$

Each angle of a regular octagon is 135°.

b. $\frac{135°}{90°}=1.5$

Each angle of a regular octagon (in right angles) is 1.5 right angles.

4. a. Textbook questions 1, 3, 5, 11, 14, 20, and 26 of "Practice," pp. 442 and 443.

1. The interior angle adjacent to the exterior angle of 66° is $180°-66°=114°$.
   The interior angle adjacent to the exterior angle of 72° is $180°-72°=108°$.

The sum of the interior angles of a hexagon is

$$\begin{aligned}180°(n-2)&=180°(6-2) \quad \longleftarrow n=6\\&=180°(4)\\&=720°\end{aligned}$$

$$\begin{aligned}\therefore\ 2(\angle 1)+105°+119°+108°+114°&=720°\\2(\angle 1)+446°&=720°\\2(\angle 1)&=274°\\\angle 1&=137°\end{aligned}$$

Section 2: Activity 4 (continued)

3. The sum of the interior angles of a pentagon is

$$\begin{aligned} 180°(n-2) &= 180°(5-2) \quad \longleftarrow n=5 \\ &= 180°(3) \\ &= 540° \end{aligned}$$

$$\begin{aligned} \therefore\ \angle 1 + 105° + (180° - 62°) + (180° - 75°) + (180° - 88°) &= 540° \\ \angle 1 + 105° + 118° + 105° + 92° &= 540° \\ \angle 1 + 420° &= 540° \\ \angle 1 &= 120° \end{aligned}$$

5.

$$\begin{aligned} 180°(n-2) &= 180°(11-2) \\ &= 180°(9) \\ &= 1620° \end{aligned}$$

$$\begin{aligned} 2n-4 &= 2(11)-4 \\ &= 22-4 \\ &= 18 \text{ right angles} \end{aligned}$$

11.

$$\begin{aligned} 180°(n-2) &= 1620° \\ n-2 &= 9 \\ n &= 11 \end{aligned}$$

The polygon has 11 sides.

14.

$$\begin{aligned} \frac{180°(n-2)}{n} &= 144° \\ 180°n - 360° &= 144°n \\ 180°n - 144°n &= 360° \\ 36°n &= 360° \\ n &= 10 \end{aligned}$$

The regular polygon has 10 sides.

20. The sum of the exterior angles is 360°.

$$\therefore\ 15°n = 360°$$
$$n = 24$$

The polygon has 24 sides.

26. $$\frac{180°(n-2)}{n} = \frac{180°(40-2)}{40}$$
$$= \frac{180°(38)}{40}$$
$$= 171°$$

Each interior angle is 171°, and each exterior angle is $180° - 171° = 9°$.

b. Textbook questions 33, 34.a., and 35 of "Applications and Problem Solving," p. 443

33. The polygon is a hexagon. Therefore, the sum of the interior angles is

$$\begin{aligned}180°(n-2) &= 180°(6-2)\\ &= 180(4)\\ &= 720°\end{aligned}$$

$$\begin{aligned}\therefore\ (3x-5)+(2x-5)+(3x-10)+(2x+10)+(3x-20)+2x &= 720°\\ 15x - 30° &= 720°\\ 15x &= 750°\\ x &= 50°\end{aligned}$$

$$\begin{aligned}\therefore\ \angle P &= 3x-5\\ &= 3(50°)-5\\ &= 145°\end{aligned} \qquad \begin{aligned}\angle Q &= 2x-5\\ &= 2(50°)-5\\ &= 95°\end{aligned} \qquad \begin{aligned}\angle R &= 3x-10\\ &= 3(50°)-10\\ &= 140°\end{aligned}$$

$$\begin{aligned}\angle S &= 2x+10\\ &= 2(50°)+10\\ &= 110°\end{aligned} \qquad \begin{aligned}\angle T &= 3x-20\\ &= 3(50°)-20\\ &= 130°\end{aligned} \qquad \begin{aligned}\angle U &= 2x\\ &= 2(50°)\\ &= 100°\end{aligned}$$

Section 2: Activity 4 (continued)

34. **a.** The sum of the interior angles of a quadrilateral is 360°.

$$\begin{aligned}\therefore\ \angle 1+40^\circ+75^\circ+30^\circ&=360^\circ\\ \angle 1+145^\circ&=360^\circ\\ \angle 1&=215^\circ\end{aligned}$$

35. **a.** $\angle ABF=\angle FBE=\angle EBD=\angle DBC$ because all are inscribed angles subtended by equal arcs.

$$\begin{aligned}\angle ABF&=\frac{1}{4}\angle ABC\\ &=\frac{1}{4}\times\frac{180^\circ(n-2)}{n}\\ &=\frac{1}{4}\times\frac{180^\circ(6-2)}{6}\\ &=\frac{1}{4}\times 120^\circ\\ &=30^\circ\end{aligned}$$

The measure of $\angle ABF$ is 30°.

b. $\angle ABD=3\angle ABF$

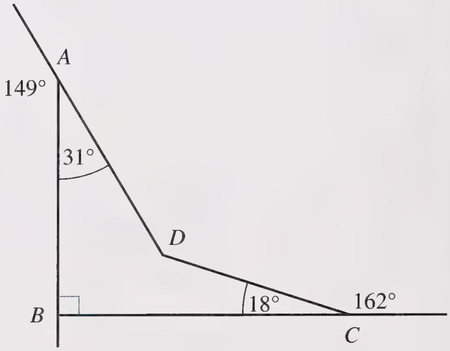
$$\begin{aligned}&=3(30^\circ)\\ &=90^\circ\end{aligned}$$

The measure of $\angle ABD=90^\circ$.

5. For the concave quadrilateral drawn, there are only three exterior angles, in the usual sense.

$$90^\circ+149^\circ+162^\circ\neq 360^\circ$$

Therefore, the exterior angles of a concave polygon do not add up to 360°.



Section 2: Follow-up Activities

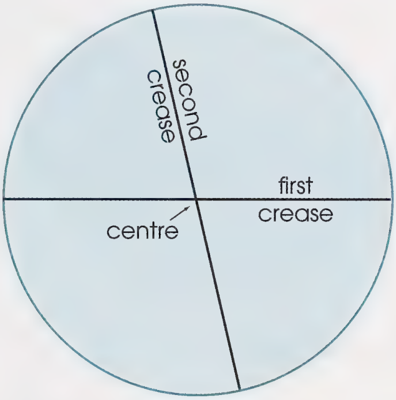
Extra Help

1. The fold made in Step 3 divided the chord in half because point *A* fell on point *B*. The angle is 90° because the chord was folded on itself.

2. The following chord theorems are illustrated by this activity:

    - *The perpendicular bisector of a chord contains the centre of the circle.*
    - *The perpendicular bisectors of two non-parallel chords intersect at the centre of the circle.*
    - *The perpendicular from the centre of a circle to a chord bisects the chord.*
    - *The line segment joining the centre of a circle and the midpoint of a chord is perpendicular to the chord.*

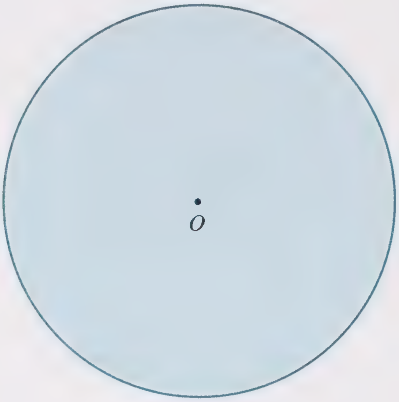
3. Fold the circle in half once. The crease is a diameter. Fold the circle in half a second time along a different diameter. The centre of the circle is at the point of intersection of the two diameters.



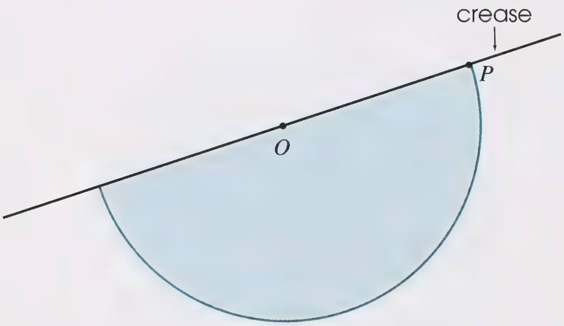
4. $\angle APB$ is a right angle, because when you make the second fold, you bisect a straight angle.

Extra Help (continued)

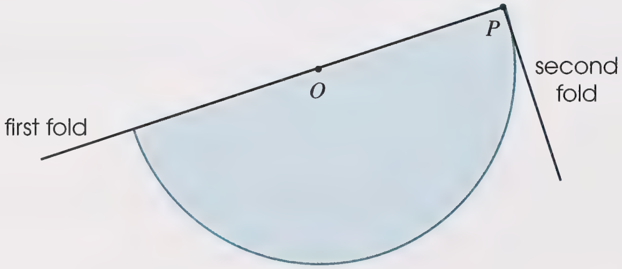
5. **Step 1:** Draw a circle on a sheet of paper, and label the centre with point *O*.



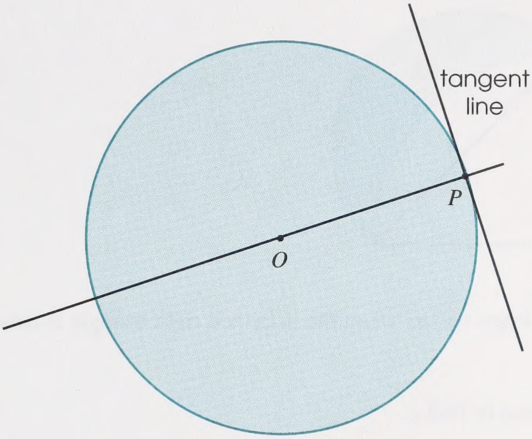
Step 2: Mark point *P* on the circle. Point *P* will be the point of tangency. Then fold the circle along line *OP*.



Step 3: Simply fold the crease back on itself at point *P*.

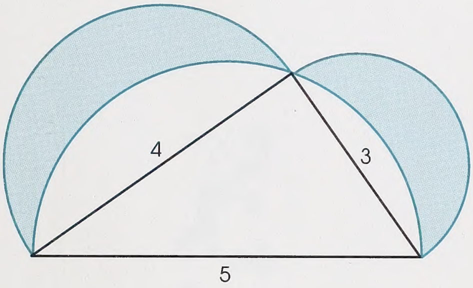


When you unfold the sheet, you will see the tangent line created by the second fold.



Enrichment

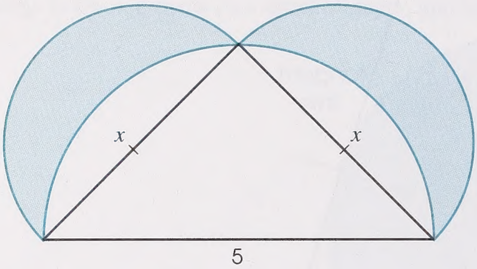
1.



$$\text{Area of the lunes} = \frac{1}{2}\ \text{base} \times \text{height}$$
$$= \frac{1}{2} \times 4 \times 3$$
$$= 6$$

Enrichment (continued)

2.



The maximum area for the lunes occurs when the inscribed right triangle is isosceles. Let x be the length of one leg.

Use the Pythagorean Theorem to find x.

$$\begin{aligned} x^2 + x^2 &= 5^2 \\ 2x^2 &= 25 \\ x^2 &= 12.5 \\ x &= \sqrt{12.5} \end{aligned}$$

Next, find the area of the lunes.

$$\begin{aligned} A &= \frac{1}{2}\text{base} \times \text{height} \\ &= \frac{1}{2}x(x) \\ &= \frac{x^2}{2} \\ &= \frac{12.5}{2} \\ &= 6.25 \end{aligned}$$

Notice this is 0.25 larger than the answer in question 1.

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